Examlet 4a **Foundations of Advanced Math** **4/12/24**

1. Consider the relation $\sim$ on $\mathbb{Z}$ defined by $x \sim y \Leftrightarrow x - y \equiv_3 3$. Determine whether $\sim$ is an equivalence relation.

$x-y \equiv_3 3 \Rightarrow 3 \mid 3-(x-y) \Rightarrow 3+y-x = 3n$ for some $n \in \mathbb{Z}$

$\Rightarrow y-x = 3n-3 = 3(n-1) = 3m$ with $m \in \mathbb{Z}$ by closure.

Rewritten: $x \sim y \Leftrightarrow y-x = 3m$ with $m \in \mathbb{Z}$ Nice!

Reflexive: For any x, $x-x = 0 = 3m$ with $m \in \mathbb{Z}$. Therefore, $x \sim x$ and $\sim$ is a reflexive relation.

Symmetric: $x \sim y \Rightarrow y-x = 3m \Rightarrow -(y-x) = -3m \Rightarrow x-y = 3(-m) = 3p$ with $p \in \mathbb{Z} \Rightarrow y \sim x$. Since $x \sim y \Rightarrow y \sim x$, $\sim$ is a symmetric relation.

Transitive: Let $x \sim y$ and $y \sim z$ so $y-x = 3n$ and $z-y = 3m$ with $n, m \in \mathbb{Z}$. Adding these statements, $(y-x)+(z-y) = 3n+3m \Rightarrow z-x = 3(n+m) = 3p$ with $p \in \mathbb{Z}$ $\Rightarrow x \sim z$. Since $x \sim y \wedge y \sim z \Rightarrow x \sim z$, $\sim$ is a transitive relation.

Since $\sim$ is reflexive, symmetric, and transitive, $\sim$ is an equivalence relation. $\square$

2. Let $S = \{a, b, c, d, e\}$, and let $\sim = \{(a,a), (b,b), (b,d), (b,e), (c,c), (d,b), (d,d), (d,e), (e,b), (e,d), (e,e)\}$

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(a) Give the equivalence classes of $\sim$.

$[a] = \{a\}$

$[b] = \{b, d, e\}$

$[c] = \{c\}$

$[d] = \{b, d, e\}$

$[e] = \{b, d, e\}$

or

$[a] = \{a\}$

$[b] = [d] = [e] = \{b, d, e\}$

$[c] = \{c\}$

Great

(b) Give the partition associated with $\sim$.

$P = \{\{a\}, \{b, d, e\}, \{c\}\}$

3. Let S be a set and Π a partition of S. Let $\sim$ be a relation on S defined by $a \sim b \Leftrightarrow \exists P \in \Pi$ for which $a, b \in P$.

(a) Show $\sim$ is a reflexive relation.

Take any $a \in S$. Well, since Π is a partition of S, it contains non-empty pairwise disjoint subsets of S whose union is all of S. So, there must $\exists P \in \Pi$ where $a \in P$, or $a, a \in P$.

$\therefore \forall a \in S$, $a \sim a$ and $\sim$ is reflexive

(b) Show $\sim$ is a symmetric relation.

Consider $a \sim b$. So, $\exists P \in \Pi$ where $a, b \in P$. And, that means $b, a \in P$ as well

$\therefore \forall a, b \in S$, $a \sim b \Rightarrow b \sim a$ and $\sim$ is symmetric

✓

(c) Show $\sim$ is a transitive relation.

Consider $a \sim b \land b \sim c$. So, $\exists P_1 \in \Pi$ where $a, b \in P_1$ and $\exists P_2 \in \Pi$ where $b, c \in P_2$. However, all $P \in \Pi$ are pairwise disjoint, and $P_1 \cap P_2 \neq \emptyset$, so that means $P_1 = P_2$.

Great

$\therefore \exists P \in \Pi$ where $a, b, c \in P$

$\forall a, b, c \in S$, $a \sim b \land b \sim c \Rightarrow a \sim c$ and $\sim$ is transitive

$f(x)=f(y) \Rightarrow x=y$

f is a relation by definition

4. Regarding the function $f : A \to B$ as a subset of $A \times B$,

(a) State the definition of f being injective.

The function f is injective iff

$\forall x, y \in A$, if $(x,z) \in f$ and $(y,z) \in f$, then

$x = y$.

Good

(b) State the definition of f being surjective.

The function f is surjective iff

$\forall b \in B \; \exists a \in A$ such that $(a,b) \in f$.

Great

5. Call two vertices v_1 and v_2 in a graph G **barely connected** iff there exists a walk from v_1 to v_2, but there exists an edge in G such that if that edge were removed, then there no longer exists a walk from v_1 to v_2. Determine whether the relation of being barely connected is reflexive, symmetric, and transitive.

- reflexive: take a graph with a single vertex v_0. A vertex cannot have an edge to itself, thus going against the rule of "there exists an edge in G..." so $a \nsim a$, and not reflexive

- symmetric: Let $v_0 \sim v_1$. This means there exists an edge in G such that if that edge were removed, then, there no longer exists a walk from v_0 to v_1. Since edges are sets, direction doesn't matter, and we know there exists a walk from v_1 to v_0, but that there exists an edge in G such that if that edge were removed, then there no longer exists the walk. So, we know $a \sim b \Rightarrow b \sim a$ making the relation symmetric. Good

- transitive: Let $v_0 \sim v_1$ and $v_1 \sim v_2$. We know that v_0 and v_1 are not in a cycle, otherwise, they would not be barely connected. The same applies for v_1 and v_2. In knowing this, we know that there is a key edge to get from v_0 to v_1 and also one for v_1 to v_2. Since v_1 is related to both v_0 and v_2, we can combine the walks to get from v_0 to v_2. We also know that those same key edges exist such that if they were removed, then there no longer exists the walk. So, $v_0 \sim v_1 \wedge v_1 \sim v_2 \Rightarrow v_0 \sim v_2$ making it transitive

Good!