Examlet 4b **Foundations of Advanced Math** **4/12/24**

1. Consider the relation $\sim$ on $\mathbb{Z}$ defined by $x \sim y \Leftrightarrow x - y \equiv_5 3$. Determine whether $\sim$ is an equivalence relation.

Reflexive: consider the relation $\sim$ being reflexive,
meaning $x - x \equiv_5 3$
But, $\underline{x - x = 0}$ and 0 modulo 5 doesn't
have a remainder of 3. So, $\sim$ is not
reflexive, already disproving that it's an
equivalence relation because it would
have to be reflexive AND symmetric And
transitive.

Great

2. Let $S = \{a, b, c, d, e\}$, and let $\sim = \{(a,a), (a,b), (b,a), (b,b), (c,c), (d,d), (d,e), (e,d), (e,e)\}$.

(a) Give the equivalence classes of $\sim$.

$[a] = \{a, b\}$

$[b] = \{a, b\}$

$[c] = \{c\}$

$[d] = \{d, e\}$

$[e] = \{d, e\}$

OR

$[a] = [b] = \{a, b\}$

$[c] = \{c\}$

$[d] = [e] = \{d, e\}$

Good

(b) Give the partition associated with $\sim$.

$\{\{a, b\}, \{c\}, \{d, e\}\}$

3. Let S be a set and Π a partition of S. Let $\sim$ be a relation on S defined by $a \sim b \Leftrightarrow \exists P \in \Pi$ for which $a, b \in P$.

(a) Show $\sim$ is a reflexive relation.

Take $a \in S$. Since Π is a partition of S, the union of the sets in Π contains all the elements in S. Therefore, $\exists P \in \Pi$ for which $a, a \in P$ meaning $a \sim a$. Thus $\sim$ is reflexive.

(b) Show $\sim$ is a symmetric relation.

If $a \sim b$ then $\exists P \in \Pi$ for which $a, b \in P$, meaning $\exists P \in \Pi$ for which $b, a \in P$. Therefore, $a \sim b \Rightarrow b \sim a$ so $\sim$ is symmetric.

(c) Show $\sim$ is a transitive relation.

S'pose $a \sim b$ and $b \sim c$. We'll say $a, b \in P_1$ and $b, c \in P_2$. But since all sets in Π are pairwise disjoint, b can not be in two different sets, meaning $P_1 = P_2$. Therefore, $a, b, c \in P_1$ so $a \sim b \wedge b \sim c \Rightarrow a \sim c$ and $\sim$ is transitive.

Good

4. Regarding the function $f : A \to B$ as a subset of $A \times B$,

(a) State the definition of f being injective.

f is injective iff $(x_1, a), (x_2, a) \in f \Rightarrow x_1 = x_2$.

Excellent!

(b) State the definition of f being surjective.

f is surjective iff $\forall b \in B \ \exists a \in A$ such that $(a, b) \in f$.

5. Call two vertices v_1 and v_2 in a graph G **evenly connected** iff the shortest walk from v_1 to v_2 has even length (where the length of a walk is the number of edges in that walk). Determine whether the relation of being evenly connected is reflexive, symmetric, and transitive.

Reflexive

A walk from V_1 to V_2 is defined by a sequence of vertices and edges where an edge is connected to both vertices it is between in the sequence, beginning with V_1 and ending with V_2.

By this definition, a sequence from a vertex V_1 to itself with 0 edges can be considered a walk, with the only element in the sequence being V_1 itself. Since all vertices have a walk with 0 edges to themselves and $0 = 2n$, $n = 0 \in \mathbb{Z}$, the relation is reflexive.

However, if we do not consider a sequence with 0 edges a walk because standing still is not walking, then vertices with no edges attached to them cannot have walks, so the relation is not reflexive.

Symmetric

If V_1 is evenly connected to V_2, then the shortest walk's sequence can be reversed to give a walk from V_2 to V_1 with the same, even number of edges as the walk from V_1 to V_2 had. This will still be the shortest walk from V_2 to V_1 because any other walk would be the same or greater length, since this particular sequence is shortest from V_1 to V_2 (provided it is not a digraph). Therefore, the relation is symmetric.

Transitive

Excellent!

Counterexample

The shortest walk from V_1 to V_2 contains 2 edges, and the shortest walk from V_2 to V_3 contains 2 edges, making both these pairs evenly connected. However, the shortest walk from V_1 to V_3 has only one edge, so they are not evenly connected. Therefore, the relation is not transitive.