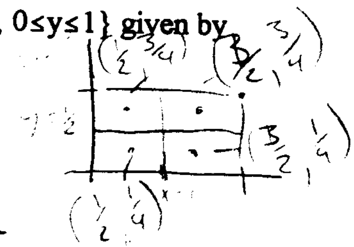
Calculus IV Exam 2 Spring 1999 3/11/99

Each problem is worth 10 points. Be sure to show all work for full credit. Please circle all answers and keep your work as legible as possible. Not intended for use as protective headgear.

-W

1. Approximate the integral $\iint_R e^{x^2+y}\,dA$ for the partition of R = {(x,y)|0≤x≤2, 0≤y≤1} given by the lines x=1 and y=½, taking (x_i^*, y_i^*) to be the center of each subrectangle.



$$= f\left(\tfrac12,\tfrac34\right)\tfrac12 + f\left(\tfrac32,\tfrac34\right)\tfrac12 + f\left(\tfrac12,\tfrac14\right)\tfrac12 + f\left(\tfrac32,\tfrac14\right)\tfrac12$$

$$= \tfrac12 e^{\frac14+\frac34} + \tfrac12 e^{\frac94+\frac34} + \tfrac12 e^{\frac14+\frac14} + \tfrac12 e^{\frac94+\frac14}$$

$$= \tfrac12\left(e^1 + e^3 + e^{\frac12} + e^{\frac52}\right)$$

2. *Set up* an integral for M_x, the moment about the x axis, of a lamina with density $\rho(x,y)=\sqrt{x^2+y^2}$ and shaped like the left half of a circle with radius 10,

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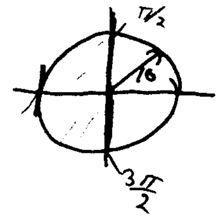
a) In rectangular coordinates

b) In polar coordinates



$10^2 = x^2 + y^2$

$y = \sqrt{10^2 - x^2}$

a) $M_x = \iint \rho y \, dy \, dx$

$= \int_{-10}^{0} \int_{-\sqrt{100-x^2}}^{\sqrt{100-x^2}} y\sqrt{x^2+y^2} \, dy \, dx$

Great

b) $M_x = \iint y \rho r \, dr \, d\theta$ $\qquad y = r\sin\theta$

$= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_{0}^{10} r^2 \sin\theta \sqrt{x^2+y^2} \, dr \, d\theta$

$= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_{0}^{10} r^3 \sin\theta \, dr \, d\theta$

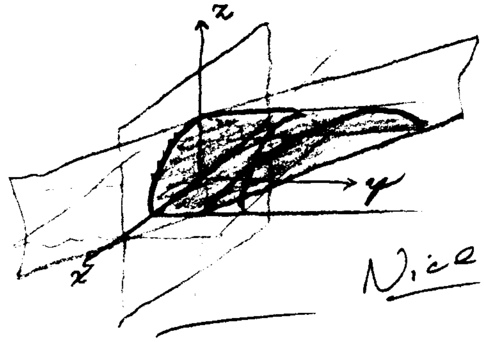
3. An artist plans to build a large abstract ice sculpture at the North pole representing the essence of polar bears. It will be shaped like the region bounded by the plane z=0, the plane y=0, the cylinder $x^2 + z^2 = 25$, and the plane $x + y = 10$. *Set up* an integral for the volume of the sculpture.

W

$$\int_{-5}^{5}\int_{0}^{10-x}\int_{0}^{\sqrt{25-x^2}} 1 \, dz\,dy\,dx$$

is the same as Yup.

$$\int_{-5}^{5}\int_{0}^{\sqrt{25-x^2}}\int_{0}^{10-x} 1 \, dy\,dz\,dx$$



Nice

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4. In computing the area of an ellipse it can be convenient to make the transformation x = a r cosθ, y = b r sinθ. Find the Jacobean for this transformation.

$x = ar\cos\theta \qquad y = br\sin\theta$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} a\cos\theta & b\sin\theta \\ -ar\sin\theta & br\cos\theta \end{vmatrix}$$

$$= (a\cos\theta)(br\cos\theta) - (-ar\sin\theta)(b\sin\theta)$$

$$= abr\cos^2\theta + abr\sin^2\theta$$

$$= abr(\cos^2\theta + \sin^2\theta)$$

$$= abr$$

Great

5. A group of happy little bunnies lives in a forest. If the total bunny population is given by the integral $\int_0^1 \int_{3y}^3 1000e^{-x^2}\,dx\,dy$,

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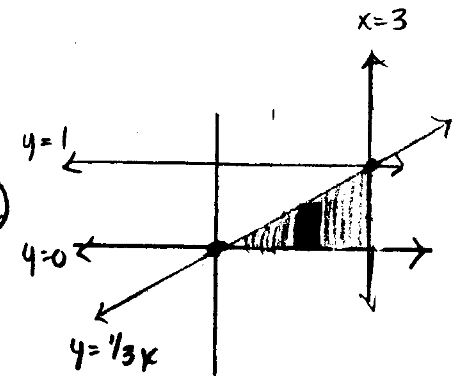
a) What function $\rho(x,y)$ gives the population density of bunnies at a point in the forest?

✓ b) What is the total bunny population in the forest, to the nearest bunny? [Hint: It may help to reverse the order of integration].

a) $\rho(x,y) = \underline{1000e^{-x^2}}$

b) $\int_0^1 \int_{3y}^3 \underline{1000e^{-x^2}}\,dx\,dy$

$x = 3$
$x = 3y$ (or $y = 1/3x$)
$y = 1$
$y = 0$



$\int_0^3 \int_0^{1/3x} 1000e^{-x^2}\,dy\,dx$

$\int_0^3 1000e^{-x^2} y \Big]_0^{1/3x} dx$

$\int_0^3 \left[\frac{1000}{3} x e^{-x^2}\right] dx$ — let $u = -x^2$ $du = -2x\,dx$

$-\frac{1}{2}\int_0^3 (-2x)\left(\frac{1000}{3}\right) x e^{-x^2}\,dx \Rightarrow -\frac{1}{2}\left(\frac{1000}{3}\right) e^{-x^2}\Big]_0^3$

$-\frac{500}{3}\left[e^{-9} - e^{-0}\right] \Rightarrow \underline{-\frac{500}{3}\left(e^{-9} - 1\right)} \approx \underline{\underline{166.64}}$

Well done

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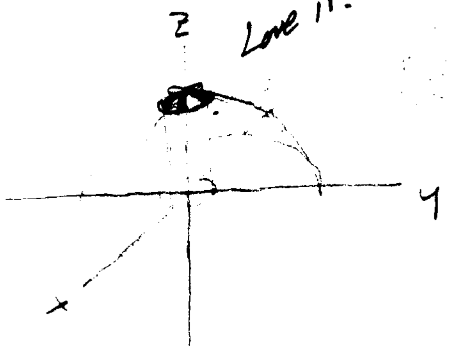
6. The OU Math Club is considering producing and marketing small crimson Math Club beanies shaped like the part of the sphere $x^2 + y^2 + z^2 = 9$ that lies within the cylinder $x^2 + y^2 = 4$. Find the surface area of such a beanie.

surface $\downarrow$ integrand $\qquad z = \sqrt{9 - x^2 - y^2} \qquad$ region $\downarrow$ limits

Love it.



$$2x + 2z\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x} = \frac{-2x}{2z} = \frac{-x}{\sqrt{9-x^2-y^2}}$$

$$2y + 2z\frac{\partial z}{\partial y} = 0 \implies \frac{\partial z}{\partial y} = \frac{-y}{\sqrt{9-x^2-y^2}}$$

$$SA = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{\frac{x^2+y^2+(9-x^2-y^2)}{9-x^2-y^2}}\,dy\,dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \frac{3}{\sqrt{9-x^2-y^2}}\,dy\,dx$$

CONVERT TO POLARS: $= 3\int_0^{2\pi}\int_0^2 (9-r^2)^{-\frac{1}{2}}\, r\,dr\,d\theta$

(U-SUB : $u = 9-r^2$; $du = -2r\,dr$; $r\,dr = \frac{du}{-2}$)

$$= -\frac{3}{2}\int_0^{2\pi}\int_{r=0}^{r=2} u^{-\frac{1}{2}}\,du\,d\theta = -3\int_0^{2\pi}\left[u^{\frac{1}{2}}\right]_{r=0}^{r=2} d\theta$$

$$= -3\int_0^{2\pi}\left[\sqrt{9-r^2}\right]_0^2 d\theta = -3\int_0^{2\pi}\left[\sqrt{5}-3\right] d\theta$$

Excellent

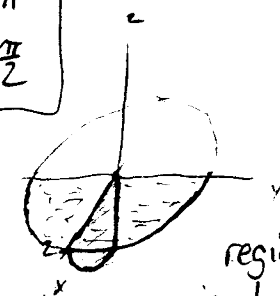
$$= -3(\sqrt{5}-3)\,\theta\Big|_0^{2\pi} = (-3\sqrt{5}+9)2\pi = (18-6\sqrt{5})\pi \approx \boxed{14.4}$$

$y = \pm\sqrt{4-x^2-z^2}$

$y^2 = 4-x^2-z^2$

$x^2+y^2+z^2=4$ sphere of $r=2$ so $\rho=2$

$0<\rho<2$
$\frac{\pi}{2}<\phi<\pi$
$-\frac{\pi}{2}<\theta<\frac{\pi}{2}$



region is bottom ½ hemisphere on + x axis side of z

Excellent

~~1~~. Evaluate the integral $\displaystyle\int_0^2\int_{-\sqrt{4-x^2}}^{0}\int_{-\sqrt{4-x^2-z^2}}^{\sqrt{4-x^2-z^2}} \frac{1}{\sqrt{\cancel{x^2+y^2+z^2}}}\,dy\,dz\,dx$. ρ^2

W

revised

$$\int_0^2\int_{\frac{\pi}{2}}^{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{\rho^2}\,\rho^2\sin\phi\, d\theta\, d\phi\, d\rho =$$

$$\int_0^2\int_{\frac{\pi}{2}}^{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\phi\, d\theta\, d\phi\, d\rho = \int_0^2\int_{\frac{\pi}{2}}^{\pi} \theta\sin\phi\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\phi\, d\rho =$$

$$\frac{\pi}{2}-\left(-\frac{\pi}{2}\right) = \frac{\pi}{2}+\frac{\pi}{2} = \frac{2\pi}{2} = \pi$$

$$\pi\int_0^2\int_{\frac{\pi}{2}}^{\pi} \sin\phi\, d\phi\, d\rho = \pi\int_0^2 -\cos\phi\Big|_{\frac{\pi}{2}}^{\pi} d\rho = \pi\int_0^2 d\rho = \pi(\rho)\Big|_0^2 = \boxed{2\pi}$$

$$-(-1)-(-(0)) = 1$$

Great Job

8. **Biff is a calculus student at O.S.U. who's having some trouble with integrating in spherical coordinates. He's just set up an integral $\int_{-5}^{5}\int_{0}^{2\pi}\int_{-\pi}^{\pi} 1\, d\phi\, d\theta\, d\rho$ for the volume of a certain region. Is there anything you can suggest Biff might need to correct about his work?**

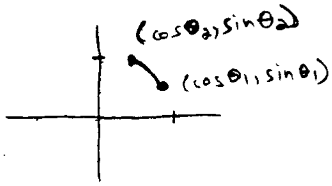
W

Biff will probably want to set up the integral as

$$\int_0^5 \int_0^{2\pi} \int_0^{\pi} \rho^2 \sin\phi \, d\phi\, d\theta\, d\rho$$

If we look at Biff's limits for ϕ we see that he has gone from $-\pi$ (which corresponds to the negative z-axis) to π (which also corresponds to the negative z-axis). We normally only want to go from 0 to π with ϕ, especially when the limits on θ are from 0 to 2π. The reason for this is that Biff's limits cause us to count the volume twice because we are rotating a full circle 2π radians which traces the same sphere twice. This same problem exists with Biff's limits for ρ. Even with the correct limits for ϕ, we are still tracing a full circle + then rotating it 2π radians around the z-axis, which again gives us two identical spheres. Correcting the limits for ρ to 0 to 5 alleviates this problem. So, we can see that with Biff's limits alone we would get a volume 4 times what it should be. In essence we are tracing two circles + then rotating each circle so that it traces two spheres. So, we end up calculating the volume of 4 such spheres. Lastly, Biff forgot to include the term which offsets the spherical coordinates (due to the transformation betwee rectangular + spherical).

Absolutely great



9. Let s be any arc of the unit circle lying entirely in the first quadrant. Let A be the area of the region lying below s and above the x-axis and let B be the area of the region lying to the right of the y-axis and to the left of s. Prove that $A + B$ depends only on the arc length, and not on the position of s. [Hints: Sketch yourself a picture and label the points at the beginning and end of the arc s as $(\cos\theta_1, \sin\theta_1)$ and $(\cos\theta_2, \sin\theta_2)$. Setting up integrals for A and B is worth half the points. In working them out, the formulas $\int\sqrt{1-x^2}\,dx = \frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\arcsin x$ and $\int\sqrt{1-x^2}\,dx = \frac{x}{2}\sqrt{1-x^2} - \frac{1}{2}\arccos x$ might be helpful.]

W

$y = \sqrt{1-x^2}$

$$A = \int_{\cos\theta_2}^{\cos\theta_1} \sqrt{1-x^2}\,dx = \left(\frac{x}{2}\sqrt{1-x^2} - \frac{1}{2}\cos^{-1}x\right)\Big|_{\cos\theta_2}^{\cos\theta_1} = \left(\frac{\cos\theta_1}{2}\sin\theta_1 - \frac{1}{2}\theta_1\right) - \left(\frac{\cos\theta_2}{2}\sin\theta_2 - \frac{1}{2}\theta_2\right)$$

$$B = \int_{\sin\theta_1}^{\sin\theta_2} \sqrt{1-y^2}\,dy = \left(\frac{y}{2}\sqrt{1-y^2} + \frac{1}{2}\sin^{-1}y\right)\Big|_{\sin\theta_1}^{\sin\theta_2} = \left(\frac{\sin\theta_2}{2}\cos\theta_2 + \frac{1}{2}\theta_2\right) - \left(\frac{\sin\theta_1}{2}\cos\theta_1 + \frac{1}{2}\theta_1\right)$$

$$A+B = \frac{\sin\theta_1\cos\theta_1}{2} - \frac{\sin\theta_1\cos\theta_1}{2} - \frac{\sin\theta_2\cos\theta_2}{2} + \frac{\sin\theta_2\cos\theta_2}{2} - \frac{1}{2}\theta_1 - \frac{1}{2}\theta_1 + \frac{1}{2}\theta_2 + \frac{1}{2}\theta_2$$

$$= \theta_2 - \theta_1$$

$$\Rightarrow A+B = \Delta\theta$$

which tells me that it only depends on arc length

Very nicely done

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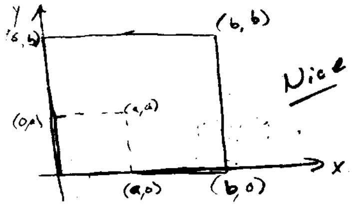
10. The region created by removing a smaller square from one corner of a larger square is called a square gnommon. If a lamina with uniform density and shaped like a square gnommon is created by removing a square with side length *a* from a square with side length *b*, where will the center of mass of the lamina lie?



Nice

density = k

$$m = \int_0^b\int_0^b k\,dx\,dy - \int_0^a\int_0^a k\,dx\,dy$$
$$= k\int_0^b [x]_0^b\,dy - k\int_0^a [x]_0^a\,dy$$
$$= k\int_0^b b\,dy - k\int_0^a a\,dy$$
$$= k[by]_0^b - k[ay]_0^a$$
$$= k(b^2 - a^2)$$

$$M_y = \int_0^b\int_0^b kx\,dx\,dy - \int_0^a\int_0^a kx\,dx\,dy$$
$$M_y = k\left(\tfrac{1}{2}\int_0^b [x^2]_0^b\,dy - \tfrac{1}{2}\int_0^a [x^2]_0^a\,dy\right)$$
$$M_y = \frac{k}{2}\left(\int_0^b b^2\,dy - \int_0^a a^2\,dy\right)$$
$$M_y = \frac{k}{2}\left([b^2 y]_0^b - [a^2 y]_0^a\right)$$
$$M_y = \frac{k}{2}(b^3 - a^3)$$

$$M_x = \int_0^b\int_0^b ky\,dx\,dy - \int_0^a\int_0^a ky\,dx\,dy$$
$$= k\left(\int_0^b [xy]_0^b\,dy - \int_0^a [xy]_0^a\,dy\right)$$
$$= k\left(\int_0^b by\,dy - \int_0^a ay\,dy\right)$$
$$= \frac{k}{2}\left([by^2]_0^b - [ay^2]_0^a\right)$$
$$= \frac{k}{2}(b^3 - a^3)$$

$$\bar{x} = \frac{M_y}{m} = \frac{\frac{k}{2}(b^3-a^3)}{k(b^2-a^2)} = \frac{1}{2}\left(\frac{b^3-a^3}{b^2-a^2}\right)$$

$$\bar{y} = \frac{M_x}{m} = \frac{\frac{k}{2}(b^3-a^3)}{k(b^2-a^2)} = \frac{1}{2}\left(\frac{b-a^3}{b-a}\right)$$

$$\left(\frac{1}{2}\left(\frac{b^3-a^3}{b^2-a^2}\right), \frac{1}{2}\left(\frac{b^3-a^3}{b^2-a^2}\right)\right)$$

Great

Extra Credit (5 points possible):

Depending on the values of *a* and *b*, the center of mass of the lamina described in problem 10 may or may not lie within the gnommon itself. For what values of *a* and *b* will the center of mass fall within the gnommon?

+4

$$a > \frac{1}{2}\left(\frac{b^3-a^3}{b^2-a^2}\right)$$
$$2a > \frac{b^3-a^3}{b^2-a^2}$$
$$2ab^2 - 2a^3 > b^3 - a^3$$
$$2ab^2 > b^3 + a^3$$
$$2a > \frac{b^3+a^3}{b^2}$$
$$2a > b - \frac{a^3}{b^2}$$

It seems to be when a is greater than 61.8% of b. This was found using the ever popular guess and test method. Oh, how we've missed it.

It goes a long way!