Problem Set 2 Key

#1

To find the highest and lowest points on the ellipse:

$\nabla f = \lambda \nabla g + \mu \nabla h$

The highest and lowest points will be on the z-axis so.

$f(x,y,z) = z \qquad g(x,y,z) = x^2 + y^2 - z = 0 \qquad h(x,y,z) = x + y + z = 24$

$x: \; 0 = \lambda(2x) + \mu(1)$

$-\mu = 2\lambda x$

$\cancel{2\lambda} y = \cancel{2\lambda} x$

$y = x$

$y: \; 0 = \lambda(2y) + \mu(1)$

$-\mu = 2\lambda y$

$z: \; 1 = \lambda(-1) + \mu(1)$

$\lambda + 1 = \mu$

To find values for x, y and z:

$x^2 + y^2 - z = 0$

$2x^2 - z = 0$

$2x^2 - 24 - 2x = 0$

$x^2 + x = 12$

$x^2 + x - 12 = 0$

$(x+4)(x-3)$

$x = -4, 3$

$y = x$ so

$y = -4, 3$

$x + y + z = 24$

$x + x + z = 24$

$2x + z = 24$

$z = 24 - 2x$

$z = 24 - 2(3) = 24 - 6 = 18$

$z = 24 - 2(-4) = 32$

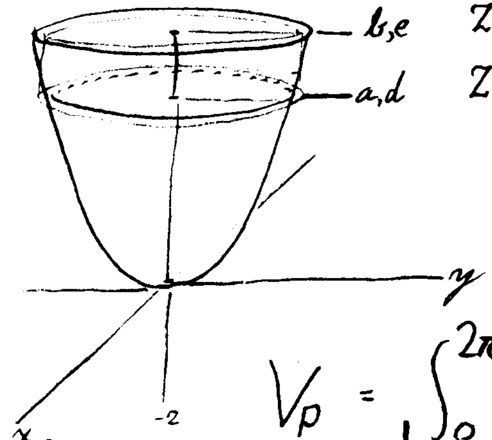
so, $(-4, -4, 32)$ and $(3, 3, 18)$

Great

$\boxed{\text{The minimum point is } (3,3,18) \text{ and the maximum point is } (-4,-4,32)}$

We know which point is the min and which is the max by looking at the z values. The largest z is the max and the smallest z is the min.

2.



$z = cx^2 + cy^2$ paraboloid

$z = cr^2$

V_p = volume of fustrum

V_c = volume of approx. cylinder

a, b = radii

d, e = heights

$$V_p = \int_0^{2\pi}\int_0^{b} cr^2 r\,dr\,d\theta - \int_0^{2\pi}\int_0^{a} cr^2 r\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{1}{4} c b^4\,d\theta - \int_0^{2\pi} \frac{1}{4} c a^4\,d\theta$$

$$\boxed{V_p = \frac{c\pi}{2}\left[b^4 - a^4\right]}$$

$d = ca^2 \;;\; e = cb^2$

$$cr^2 = z_{\frac{1}{2}} = \left(\frac{d+e}{2}\right) = \frac{c[a^2+b^2]}{2}$$

$$r^2 = \frac{a^2+b^2}{2}$$

$$V_c = \underbrace{\left[\frac{c(a^2+b^2)}{2}\cdot\pi\right]}_{\text{Area } (r^2\cdot\pi)} \times \underbrace{\left[(e-d)\right]}_{\text{Height}}$$

$$V_c = \frac{\pi}{2}(a^2+b^2)(ca^2 - cb^2)$$

$$\boxed{V_c = \frac{c\pi}{2}\left[a^4 - b^4\right]}$$

Good

Problem 3

Maclaurin also showed that the difference between a frustum of a right circular cone and the corresponding cylinder is one-fourth the volume of a similar cone, with the same height as the frustum and with diameter one-half the difference between the upper and lower diameters of the frustum. Use a double integral to express the volume of a frustum of a right circular cone and show why this is true.

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Volume of frustum:

$$\int_0^{2\pi}\int_0^{b/k}(b-rk)\,r\,dr\,d\theta - \int_0^{2\pi}\int_0^{a/k}(a-rk)\,r\,dr\,d\theta =$$

$$\int_0^{2\pi}\int_0^{b/k}(br-r^2k)\,dr\,d\theta - \int_0^{2\pi}\int_0^{a/k}(ar-r^2k)\,dr\,d\theta =$$

$$\int_0^{2\pi}\left(\frac{br^2}{2}-\frac{r^3k}{3}\right)\Big|_0^{b/k}d\theta - \int_0^{2\pi}\left(\frac{ar^2}{2}-\frac{r^3k}{3}\right)\Big|_0^{a/k}d\theta =$$

$$\int_0^{2\pi}\left[\frac{b^3}{2k^2}-\frac{b^3}{3k^2}\right]d\theta - \int_0^{2\pi}\left[\frac{a^3}{2k^2}-\frac{a^3}{3k^2}\right]d\theta =$$

$$\int_0^{2\pi}\left(\frac{b^3}{6k^2}\right)d\theta - \int_0^{2\pi}\left(\frac{a^3}{6k^2}\right)d\theta = \frac{b^3}{6k^2}\,\theta\Big|_0^{2\pi} - \frac{a^3}{6k^2}\,\theta\Big|_0^{2\pi} =$$

$$\frac{b^3\pi}{3k^2} - \frac{a^3\pi}{3k^2} = \boxed{\frac{\pi}{3k^2}(b^3-a^3)} = \underline{V_f}$$

Volume of approximating cylinder:

$$\pi(b-a)\left(\frac{b+a}{2k}\right)^2 = \pi(b-a)\left(\frac{b^2+2ab+a^2}{4k}\right) =$$

$$\frac{\pi}{4k^2}(b^3+2ab^2+a^2b-ab^2-2a^2b-a^3) =$$

$$\boxed{\frac{\pi}{4k^2}(b^3+ab^2-a^2b-a^3)} = \underline{V_c}$$

Volume of the similar cone:

$$V = \frac{1}{3}\pi\left(\frac{b-a}{k}\right)^2 (b-a) = \frac{\pi}{3k^2}(b^2-2ab+a^2)(b-a) =$$

$$\frac{\pi}{3k^2}(b^3-2ab^2+a^2b-ab^2+2a^2b-a^3) =$$

$$\boxed{\frac{\pi}{3k^2}(b^3-3ab^2+3a^2b-a^3)} = V_\Delta$$

Determine the difference between the approximating cylinder and frustum:

$$D = \frac{\pi}{3k^2}(b^3-a^3) - \frac{\pi}{4k^2}(b^3+ab^2-a^2b-a^3)$$

$$= \frac{4\pi}{12k^2}(b^3-a^3) - \frac{3\pi}{12k^2}(b^3+ab^2-a^2b-a^3)$$

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$$= \frac{\pi}{12k^2}b^3 - \frac{\pi}{12k^2}a^3 + \frac{3\pi}{12k^2}ab^2 - \frac{3\pi}{12k^2}a^2b$$

$$\boxed{= \frac{\pi}{12k^2}(b^3+3ab^2-3a^2b-a^3)} = V_{Diff}$$

$$\therefore \frac{1}{4}(V_\Delta) = V_{Diff}$$

Excellent

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4) a- see graph

b- this temperature distribution is likely to be caused by a wall heater at one end of a room.

c- Mathematica evaluates $\int_{-3}^{3}\int_{0}^{5}(68+6e^{-0.02x^2-0.05y^2})\,dy\,dx$

to be 2173.72, Dividing this by the area of the room $(5\cdot 6=30)$ we get

$$2173.72/30 = 72.4573 \approx \boxed{72.5}$$

Which is the average height (or temperature in degrees fahrenheit) for the function over the range specified.

Great

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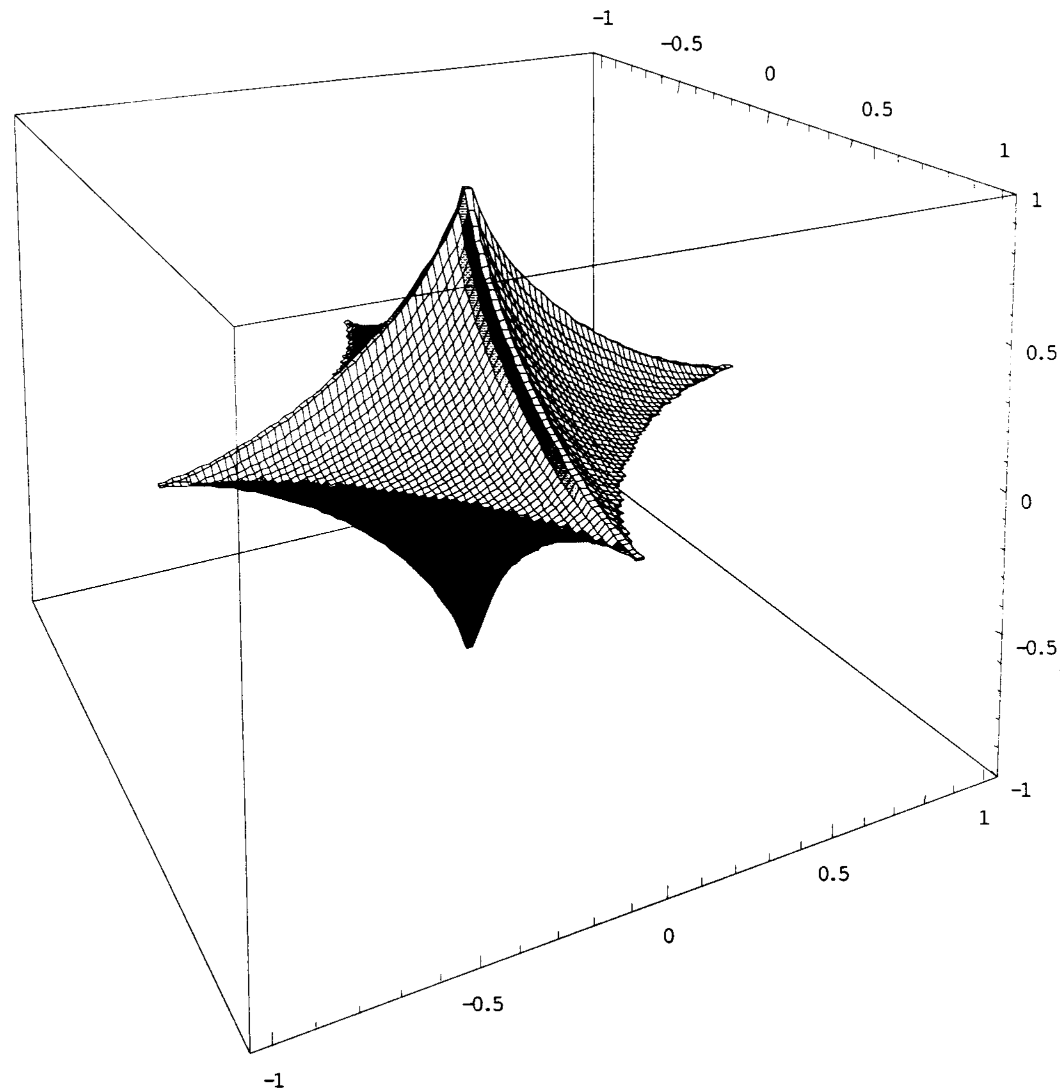
5. a) $x^{2/3} + y^{2/3} + z^{2/3} = 1$. Solve for z: $z = (1 - x^{2/3} - y^{2/3})^{3/2}$ & graph:

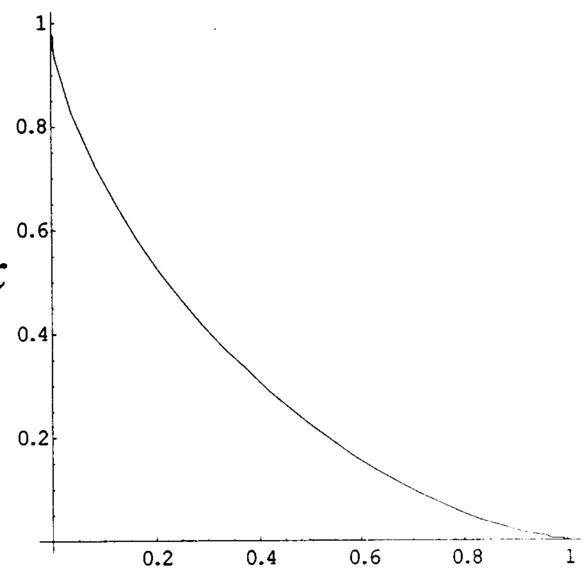


b)

Solve for x or y when $z = 0 = (1 - x^{2/3} - y^{2/3})^{3/2}$ yields $x = (1 - y^{2/3})^{3/2}$ & $y = (1 - x^{2/3})^{3/2}$.

A top down view of the first octant is shown. If you take the x values ranging from 0 to1 and the y values ranging from 0 to $(1 - x^{2/3})^{3/2}$ then you can set up the double integral as follows: (8 times this value is the volume.)

$$\int_0^1 \int_0^{(1-x^{2/3})^{3/2}} \left(1 - x^{2/3} - y^{2/3}\right)^{3/2} dy\,dx$$



c) By letting $y = \sin^3\theta*(1 - x^{2/3})^{3/2}$ and then finding $dy = 3\sin^2\theta*\cos\theta*(1 - x^{2/3})^{3/2}\, d\theta$ and getting new limits of integration, you have the following integral which can be solved:
when $y = 0$, $\theta = k\pi = 0$ (if k=0)
when $y = (1 - x^{2/3})^{3/2}$, $\theta = 2k\pi + \pi/2 = \pi/2$

$$\int_0^1 \int_0^{\frac{\pi}{2}} \left(1 - x^{\frac{2}{3}} - \left(\sin^3\theta \left(1 - x^{\frac{2}{3}}\right)^{\frac{3}{2}}\right)^{\frac{2}{3}}\right)^{\frac{3}{2}} \left(1 - x^{\frac{2}{3}}\right)^{\frac{3}{2}} 3\sin^2\theta \cos\theta \, d\theta \, dx$$

$$= \int_0^1 \int_0^{\frac{\pi}{2}} \left(1 - x^{\frac{2}{3}}\right)^3 \left(1 - \sin^2\theta\right)^{\frac{3}{2}} 3\sin^2\theta \cos\theta \, d\theta \, dx = \frac{\pi}{70}$$

and since this is the volume of one octant, the resulting volume for the entire solid is : $\dfrac{8*\pi}{70} = \dfrac{4\pi}{35} \approx .359$