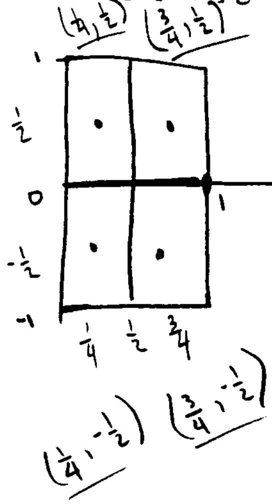
Calculus IV Exam 2 Summer 2000 7/7/2000

Each problem is worth 10 points. Show all work for full credit. Please circle all answers and keep your work as legible as possible. May not be safe for use a drinking water.

1. Calculate the double Riemann sum of the function $f(x,y) = 3x + y^2$ for the partition of $R=[0,1]\times[-1,1]$ given by $x=\frac{1}{2}$ and $y=0$, using (x_{ij}^*, y_{ij}^*) = the midpoint of R_{ij}.

W



$\Delta x \Delta y = \frac{1}{2}$

$= \frac{1}{2}\left[f\left(\frac{1}{4},\frac{1}{2}\right) + f\left(\frac{3}{4},\frac{1}{2}\right) + f\left(\frac{1}{4},-\frac{1}{2}\right) + f\left(\frac{3}{4},-\frac{1}{2}\right)\right]$

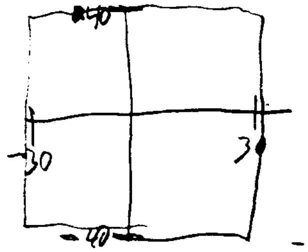
$= \frac{1}{2}\left[3\left(\frac{1}{4}\right) + \left(\frac{1}{2}\right)^2 + 3\left(\frac{3}{4}\right) + \left(\frac{1}{2}\right)^2 + 3\left(\frac{1}{4}\right) + \left(-\frac{1}{2}\right)^2 + 3\left(\frac{3}{4}\right) + \left(-\frac{1}{2}\right)^2\right]$

$= \frac{1}{2}\left[\frac{3}{4} + \frac{1}{4} + \frac{9}{4} + \frac{1}{4} + \frac{3}{4} + \frac{1}{4} + \frac{9}{4} + \frac{1}{4}\right]$

$= \frac{1}{2}\left[\frac{28}{4}\right] = \boxed{\frac{7}{2}}$

10

2. The Oklahoma legislature has decided that instead of giving graduate teaching assistants a long-deserved and much-needed pay increase, they're going to use the funds to tear down the Physical Sciences building and replace it with an even goofier structure with a base shaped like the rectangle [-30,30]×[-40.40] in the xy-plane and with the top shaped like the portion of the paraboloid $z=2500-x^2-y^2$. *Set up* a double or triple integral for the volume of this new building.



$$\boxed{\int_{-30}^{30}\int_{-40}^{40}\int_{0}^{2500-x^2-y^2} 1 \, dz\,dy\,dy}$$

Rectangular base means constants for xy bounds.

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4. Archeologists find a coprolite shaped like the solid bounded by the paraboloid $z=6-x^2-y^2$ and the plane $z=0$. As part of their analysis they drill out a hole do~~wn the center (i.e.,~~ along the z axis) of radius 1. *Set up* a multiple integral for the volume of the remaining solid.

$$\int_0^{2\pi}\int_{-\cancel{x}}^{\cancel{x}}\int_0^{6-r^2} 1\cdot r\cdot dz\,dr\,d\theta$$

Top view?

This would almost give the volume of the hole (rather than the remaining solid) but that negative value for r is just a bad idea.

4. Archeologists find a coprolite shaped like the solid bounded by the paraboloid $z=6-x^2-y^2$ and the plane $z=0$. As part of their analysis they drill out a hole down the center (i.e., along the z axis) of radius 1. *Set up* a multiple integral for the volume of the remaining solid.

limit

$z = 6-x^2-y^2$

$z = 0 | z$

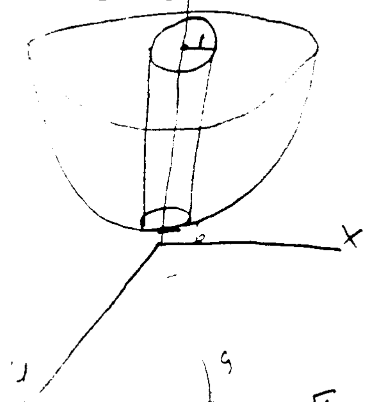
$x^2+y^2=1$

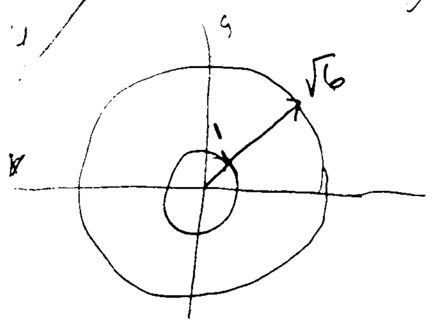
$z = 6-x^2-y^2$

$x^2+y^2 = 6$

$r^2 = 6$

$r = \sqrt{6}$





Great

$$\int_0^{2\pi}\int_1^{\sqrt{6}}\int_0^{6-r^2} r\,dz\,dr\,d\theta$$

5. For some really, really important reason you need to make a change of variable in a double integral, where the desperately needed transformation is given by x=3u-6v, y=-2u+4v. Find the Jacobian for this transformation.

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 3 & -2 \\ -6 & 4 \end{vmatrix} = 12 - (12) = 0$$

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6. Find the exact value of $\int_0^1 \int_{3y}^3 e^{x^2}\,dx\,dy$.

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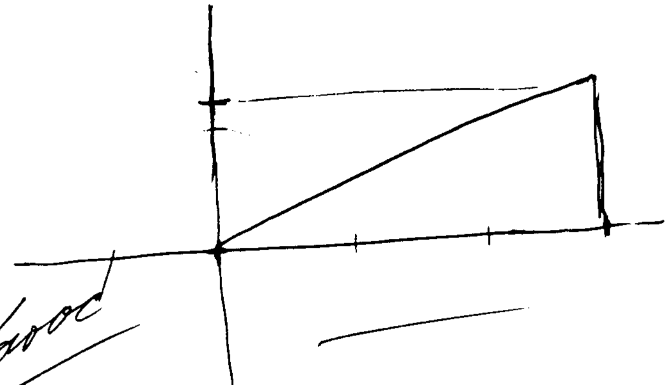
switch order of integration

$y = \frac{1}{3}x$



$$\int_0^3 \int_0^{\frac{1}{3}x} e^{x^2}\,dy\,dx$$

$$= \int_0^3 y \cdot e^{x^2}\Big|_0^{\frac{1}{3}x}\,dx = \frac{1}{3}\int_0^3 x e^{x^2}\,dx$$

Good

let $u = x^2$

$\frac{du}{dx} = 2x$

$dx = \frac{du}{2x}$

$$= \frac{1}{3}\int_{x=0}^{x=3} x e^{u}\,\frac{du}{2x} = \frac{1}{6}\int_{x=0}^{x=3} e^{u}\,du$$

$$= \frac{1}{6} e^{u}\Big|_{x=0}^{x=3} = \frac{1}{6} e^{x^2}\Big|_{x=0}^{x=3} = \frac{1}{6}\left[e^9 - 1\right]$$

7. Barbie is a Calc IV student from California. Barbie says "Multiple integrals are *hard.* I can never figure out what to put for the limit thingies. Like, it's always different and so confusing, but then I think I ~~got~~ have a plan. I, like, started just always doing in the first quarter-thingy, you know? And then I just multiply it by four if the picture is all above the, you know, the middle-thingy, or by eight if some of it is below the middle-thingy. That makes it a lot less stress."

Clearly explain, in a way that Barbie can understand, why what she's doing might be valid or invalid. Be sure to include some specifics about when it will or will not work.

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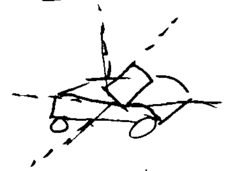
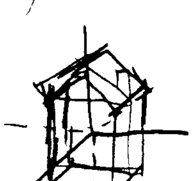
This is a horrible IDEA BARBIE. What if the thing you are integrating is Not symetrical about all axis like your Corvette?



Is Multiplying the function that grafts your car's front right quarter panel going to give you a Whole car? No it will not. you'll end up with twice the area of the engine and none of the area of the seats. And what if your car was positioned so that only the front bumper was in the Positive y axis? Multiplying by 4 will not get you anything more than a small percentage of your ride.

The Multipling by 8 idea is bad too. Say you stuck the axis so that the first floor of your Mansion was below they XY plane and bisected your house down the middle for the XZ + YZ Planes



The 8 parts of your house are non symetrical and won't Equal the integral if you multiply by 8.

Oh and Barbie, You have a plan not got a plan. And stop saying "you know" unless the person knows another person or thing. (Damn Valley Girl accents, my Girlfriend speaks The Same way

8. A partially decomposed moose carcass happens to be shaped exactly like the region bounded by $z=4-x^2$ and the planes $x+y=6$, $y-x=6$, $y=0$, and $z=0$. Find the volume of the carcass.

DAVE BARRY

I have here a newspaper item, sent in by many alert readers of the *Fairbanks Daily News-Miner*, concerning an Alaska homeowner named William Keith whose plumbing backed up. Keith called a professional, who determined that the septic system was being blocked by a dead moose.

If you suspect your plumbing has the same problem, the way to find out is to examine your septic tank and see if you spot a large dead moose. If so, remain calm, get into your car, drive away and never return. I say this because the *Fairbanks Daily News-Miner* printed a color photograph of the deceased moose being hoisted out of the hole, and it is way scarier than anything I ever saw on *The X-Files*.

Wednesday

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APRIL

10

$y = -x+6 \qquad x = -y+6$

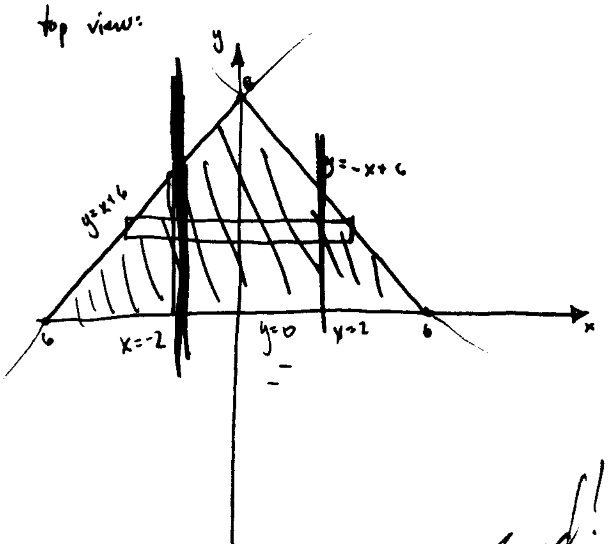
$y = x+6 \qquad x = y-6$

$y = 0$

$z = 0$

$z = 4-x^2$

top view:



$\cancel{\int_0^6\int_{y-6}^{-y+6}} 4-x^2\,dx\,dy$

$= \int_0^6 \left[4x-\frac{1}{3}x^3\right]_{y-6}^{-y+6} dy$

~~yuck~~

$2\int_0^2\int_0^{-x+6}(4-x^2)\,dy\,dx = 2\int_0^2\left[(4-x^2)y\right]_0^{-x+6}dx$

$= 2\int_0^2(x^3-6x^2-4x+24)\,dx$

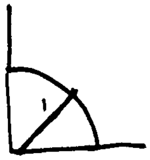
Good!

$= 2\left[\frac{1}{4}x^4-2x^3-2x^2+24x\right]_0^2$

$= 2(4-16-8+48) = 2(28) = 56 \text{ units}^3$

9. Find the centroid of the portion of a sphere of radius 1 which lies in the first octant.

$x^2+y^2+z^2=1 \quad \rho^2=1 \quad \rho=1$



$m = \iiint_E \rho f(x,y,z)\, dV = \iiint_E 1\cdot f(\rho,\theta,\phi)\,\rho^2 \sin\phi \, d\rho\, d\theta\, d\phi$

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$= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi = \int_0^{\pi/2}\int_0^{\pi/2} \frac{\rho^3}{3}\sin\phi \Big|_0^1 \, d\theta\, d\phi$

$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{1}{3}\sin\phi \, d\theta\, d\phi = \int_0^{\frac{\pi}{2}} \frac{\theta}{3}\sin\phi \Big|_0^{\frac{\pi}{2}} d\phi = \int_0^{\pi/2} \frac{\pi}{6}\sin\phi \, d\phi$

$= \frac{\pi}{6}\left|-\cos\phi\right|_0^{\pi/2} = \frac{\pi}{6}\left(-0-(-1)\right) = \frac{\pi}{6} = m$

$\bar{x} = \frac{M_{yz}}{m}$

$\bar{y} = \frac{M_{xz}}{m}$

$\bar{z} = \frac{M_{xy}}{m}$

$M_{yz} = \iiint_E x\, f(x,y,z)\, dV$

Because the surface is symmetric, the $\bar{x}$, $\bar{y}$ and $\bar{z}$ will all be the same

$z = \rho\cos\phi$

$M_{xy} = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^1 \rho^3 \sin\phi\cos\phi \, d\rho\, d\phi\, d\theta$

$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{\rho^4}{4}\sin\phi\cos\phi \Big|_0^1 d\phi\, d\theta = \int_0^{\pi/2}\int_0^{\pi/2} \frac{1}{4}\sin\phi\cos\phi \, d\phi\, d\theta$

$= \frac{1}{4}\int_0^{\pi/2}\int_0^{\pi/2} \sin\phi\cos\phi \, d\phi\, d\theta = \frac{1}{4}\int_0^{\pi/2} \frac{1}{2}\, d\theta = \frac{1}{4}\left|\frac{\theta}{2}\right|_0^{\pi/2}$

$\frac{1}{4}\left|\frac{\pi}{4}-0\right| = \frac{\pi}{16}$

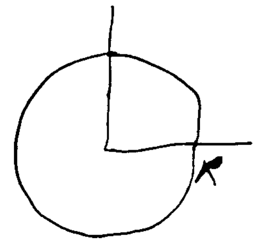
$\frac{M_{xy}}{m} = \frac{\frac{\pi}{16}}{\frac{\pi}{6}} = \frac{3}{8}$

so the centroid is located at

$\left(\frac{3}{8}, \frac{3}{8}, \frac{3}{8}\right)$

Yes!

10. Find the surface area of the portion of the plane z=ax which lies inside the cylinder $x^2+y^2=R^2$.



$$= \iint \sqrt{a^2+0^2+1}\; dy\, dx$$

$$= \int_0^{2\pi}\int_0^R \sqrt{a^2+1}\; r\, dr\, d\theta$$

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$$= \int_0^{2\pi} \sqrt{a^2+1}\; \frac{r^2}{2}\Big|_0^R\, d\theta$$

$$= \int_0^{2\pi} \sqrt{a^2+1}\; \frac{R^2}{2}\, d\theta$$

Great

$$= \frac{2\pi R^2\sqrt{a^2+1}}{2} = \underline{\pi R^2\sqrt{a^2+1}}$$