[redacted]

Exam 1 Calc IV (Math2443-003) 6/15/2001

Each problem is worth 10 points. Show all work for full credit. Please circle all answers and keep your work as legible as possible. "Slug" should not be taken as a pejorative term.

1. Find $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+xy+y^2}$ or show that the limit does not exist.

10

Set $x=0$

$$\lim_{(0,y)\to(0,0)} \frac{(0)(y)}{(0^2)+(0)(y)+y^2} = \lim_{y\to 0} \frac{0}{y^2} = 0$$

Set $y=0$

$$\lim_{(x,0)\to(0,0)} \frac{x(0)}{x^2+x(0)+0^2} = \lim_{x\to 0} \frac{0}{x^2} = 0$$

Set $y=x$

$$\lim_{(x,x)\to(0,0)} \frac{(x)(x)}{x^2+(x)(x)+x^2} = \lim_{x\to 0} \frac{x^2}{3x^2} = 1/3$$

These 3 do not equal, therefore the limit DNE.

Nice

2. Find the directional derivative of $f(x,y) = e^{x-y}$ at the point (2,1) in the direction of the vector <3,-4>.

10

① First, find $f_x(x,y)$ and $f_y(x,y)$

$f(x,y) = e^{x-y}$

$f_x(x,y) = e^{x-y}(1) = e^{x-y}$ $\quad$ $f_y(x,y) = e^{x-y}(-1) = -e^{x-y}$

Now plug in the point (2,1) into f_x and f_y

$f_x(2,1) = e^{2-1} = e \approx 2.7183$

$f_y(2,1) = -e^{2-1} = -e \approx -2.7183$

Beautifully Done!

Before we can plug things into the directional derivative equation, we must convert the vector <3,-4> into a unit vector.

$|\vec{v}| = \sqrt{3^2+(-4)^2} = 5 \qquad \vec{u} = \langle \frac{3}{5}, -\frac{4}{5} \rangle$

④ Now, plug things into the directional derivative equation: $D_{\vec{u}}f(x,y) = af_x + bf_y$

$D_{\vec{u}}f(2,1) = \frac{3}{5}e + -\frac{4}{5}(-e) = \frac{3}{5}e + \frac{4}{5}e = \frac{7}{5}e$

$$\boxed{D_{\vec{u}}f(2,1) = \frac{7}{5}e \approx 3.8056}$$

3. If we have z = f(x,y) where x = x(s,t,u), and y = y(s,t,u), write out the appropriate chain rule for $\frac{\partial z}{\partial s}$.

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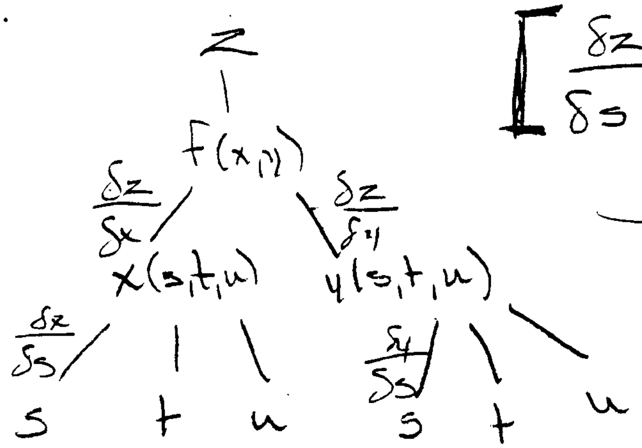


$$\left[\frac{\delta z}{\delta s} = \frac{\delta z}{\delta x} \cdot \frac{\delta x}{\delta s} + \frac{\delta z}{\delta y} \cdot \frac{\delta y}{\delta s}\right]$$

Good

4. The function $f(x,y) = 2x^2 - 4xy + y^4 + 2$ has critical points at (1,1), (0,0) and (-1,-1). Classify each of them as a maximum, minimum, saddle point, or neither.

10

$f(x,y) = 2x^2 - 4xy + y^4 + 2$

Critical Points

(1,1) (0,0) (-1,-1)

$f_x(x,y) = 4x - 4y$

$f_y(x,y) = -4x + 4y^3$

$D(x,y) = f_{xx} f_{yy} - (f_{xy})^2$

$f_{xx}(x,y) = 4$

$f_{yy}(x,y) = 12y^2$

$f_{xy}(x,y) = -4$

f_{yy}

At (1,1)

$D(1,1) = 4(12) - 16 = 32$

Excellent

Since D = 32 and 32 > 0 it is a max or min

$f_{xx}(1,1) = 4$ since 4 > 0 the point (1,1) is at a minimum!

At (0,0)

$D(0,0) = 4(0) - 16 = -16$

Since D = -16 since -16 < 0 the point (0,0) is at a saddle point!

At (-1,-1)

$D(-1,-1) = 4(12) - 16 = 32$

Since D = 32 and 32 > 0 it is a max or m

$f_{xx}(1,1) = 4$ since 4 > 0 the point (-1,-1) is at a minimum also!

5. Sammy the slug is dreaming about a beautiful cabbage leaf. If the leaf resembles the surface $z=\sqrt{9-x^2-y^2}$ and Sammy is standing at the point (1,-2,2), write an equation of the plane tangent to the cabbage at the point where Sammy is standing.

W

$$z-z_0 = f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$$

$$\boxed{(z-2)=-\tfrac{1}{2}(x-1)+(y+2)}$$

Great

$$f(x,y)=z=\sqrt{9-x^2-y^2}=(9-x^2-y^2)^{1/2}$$

$$f_x(x,y)=\tfrac{1}{2}(9-x^2-y^2)^{-1/2}(-2x)$$

$$=-\frac{x}{\sqrt{9-x^2-y^2}}$$

$$f_x(1,-2)=-\tfrac{1}{2}$$

$$f_y(x,y)=-\frac{y}{\sqrt{9-x^2-y^2}}$$

$$f_y(1,-2)=1$$

W

6. For the function g(x,y) = cos$\sqrt{x^2+y^2}$ at the point (0, π/2) find the direction in which the directional derivative is greatest and the value of that directional derivative.

gradient is direction of greatest

length of gradient is value of greatest

$g(x,y) = \cos(x^2+y^2)^{1/2}$

$g_x = -\sin(x^2+y^2)^{1/2} \cdot 1/2(x^2+y^2)^{-1/2} \cdot 2x = \underline{0}$ at $(0, \pi/2)$

$-\sin(x^2+y^2)^{1/2} \cdot (x^2+y^2)^{-1/2} \cdot x$

$g_y = -\sin(x^2+y^2)^{1/2} \cdot 1/2(x^2+y^2)^{-1/2} \cdot 2y = \underline{-1}$ at $(0, \pi/2)$

$-\sin\frac{\pi}{2} \cdot \frac{1}{2\left(\frac{\pi}{2}\right)} \cdot \pi$

$\langle 0, -1 \rangle$ ← where direction is greatest

$\sqrt{0^2+(-1)^2} = \underline{1}$

Well done

7. Jebediah is a calculus student as O.S.U. who's having some trouble with directional derivatives. Jeb says "Gosh darn it, I think I just spread a whole load of manure all over my calc test. There was this one question about those directional doo-hickys, and it wanted to know, like, if you had the directional derivative for one direction, then could you get the directional derivative for the opposite direction. I figured maybe it might be just the same, so I put that, but some girls was talking after the test and they was saying something totally different."

W

Help Jeb out by explaining (in a way he can understand!) what you can say about the directional derivatives in opposite direcctions.

Well golly, Jeb, you're purty darn close there. You see, if ya took the directional derivative for that there same point in the opposite direction, all ya'd really be doin is swappin them there signs around. So ya'd end up with the same numbers just the opposite sign! You're purty darn good, Jeb. What's say you an' me study together some time? :)

Darn purty answer!

In other words...

Jeb, you're almost right. When you take the directional derivative of a particular point in the opposite direction, you simply change the direction of the unit vector which would be indicated by the sign of each component. i.e. $\vec{u}_0 \langle 3/5, -4/5 \rangle$ as in problem 2 of this test would be $\vec{u}_f \langle -3/5, 4/5 \rangle$ in the opposite direction. Therefore, continuing to look at problem 2 with the opposite direction you get $D_{\vec{u}_f} = -3/5 e + 4/5 e = \frac{e}{5}$, which is the same numerical value as before with the opposite sign. Very quick on the uptake, Jeb. Would you like to study together some time?

Very well done!

8. Find the location of the minimum value of the function $p(x,y) = x^2 + y^2 + ax + by + c$.

$P_x = 2x + a$ $\quad P_x = 0 \Rightarrow x = -\frac{a}{2}$ $\quad \rightarrow$ critical point : $A\left(-\frac{a}{2}, -\frac{b}{2}\right)$

$P_y = 2y + b$. $\quad P_y = 0 \Rightarrow y = -\frac{b}{2}$.

$P_{xx} = 2$

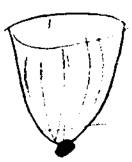
$P_{yy} = 2$. $\quad D = P_{xx} \cdot P_{yy} - P_{xy} = 2 \times 2 - 0 = 4 > 0$

W

$P_{xy} = 0$. $\quad \Rightarrow$ either max or min

$P_{xx} > 0 \Rightarrow A\left(-\frac{a}{2}, -\frac{b}{2}\right)$ is a minimum value

* As we can see from the graph; it is a paraboloid and concave. Therefore, it should have a minimum value. It's the bottom of the graph.



Very well done

9. If f(x,y) = sin x + sin y, what is the largest value D_u f(x,y) can have?

11

$f_x = \cos x$

$f_y = \cos y$

$\nabla f = \langle \cos x, \cos y \rangle$

$|\nabla f| = \sqrt{\cos^2 x + \cos^2 y}$

$= \sqrt{1^2 + 1^2}$

$= \sqrt{2}$

Yes.

10. Show that every plane tangent to the cone $x^2 + y^2 = z^2$ passes through the origin.

$$2z\frac{\partial z}{\partial x} = 2x \Rightarrow \frac{\partial z}{\partial x} = \frac{x}{z} \qquad 2z\frac{\partial z}{\partial y} = 2y \Rightarrow \frac{\partial z}{\partial y} = \frac{y}{z}$$

$$z = \frac{x_0}{z_0}(x - x_0) + \frac{y_0}{z_0}(y - y_0) + z_0$$

$$= \frac{x_0}{z_0}x - \frac{x_0^2}{z_0} + \frac{y_0}{z_0}y - \frac{y_0^2}{z_0} + z_0$$

$$= \frac{x_0}{z_0}x + \frac{y_0}{z_0}y - \frac{1}{z_0}(x_0^2 + y_0^2) + z_0 \qquad x_0^2 + y_0^2 = z_0^2$$

$$= \frac{x_0}{z_0}x + \frac{y_0}{z_0}y - \frac{1}{z_0}(z_0^2) + z_0$$

10

$$= \frac{x_0}{z_0}x + \frac{y_0}{z_0}y \boxed{- z_0 + z_0}$$

$$= \frac{x_0}{z_0}x + \frac{y_0}{z_0}y$$

Nice!

$0 = z$-intercept ∴ $x^2 + y^2 = 0$ ∴ x & y must equal 0 when $z = 0$ ∴ all tangent planes pass through the point $(0,0,0)$ which is the origin

+4

Extra Credit (5 points possible):

A function of two variables whose partial derivatives of all orders are continuous has at most three distinct second order partial derivatives, since $f_{xy}=f_{yx}$. How many distinct third partials might it have? Can you say anything similiar about fourth and higher order partials? [Hint: If nothing else, take a function like $f(x,y) = x^4y^3$ and find all of its third order partial derivatives, then look at them...]

$f_x(x,y) = 4x^3y^3$

$f_y(x,y) = \cancel{0}\ 3x^4y^2$

$f_{xx}(x,y) = 12x^2y^3$

$f_{yy}(x,y) = 6x^4y$

$f_{xy}(x,y) = 36x^2y^2$

$f_{xxx}(x,y) = 24xy^3$

$f_{yyy}(x,y) = 6x^4$

$f_{yxy} = ? = f_{xyx}$

$f_{xxy} = ?$

$f_{yyx} = ?\ f_{xxy}$

The 3rd derivative has 4 distinct partials. From there every n^{th} derivat has $n+1$ distinct partials