Exam 2 Calc IV (Math2443-003) 7/6/2001

Each problem is worth 10 points. Show all work for full credit. Please circle all answers and keep your work as legible as possible. Don't count your chickens before they hatch.

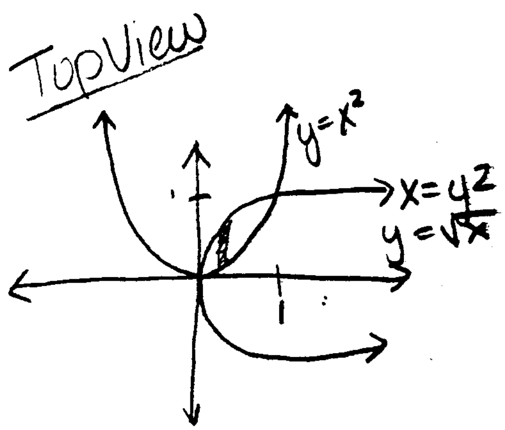
1. A team of researchers is studying a rare specices called the parallelogram newt. The researchers have determined that the average newt population in the study area is roughly given by the function f(x,y) = 200 + 4x - 12y dung beetles per square mile in a region ranging from 0 to 10 on their x axis and 0 to 15 on their y axis. According to their model, what is the total population of newts in the research area?

$$\int_0^{15}\int_0^{10} 200+4x-12y \, dx\, dy = \int_0^{15} \left[200x + 2x^2 - 12xy\right]_0^{10} dy$$

$$= \int_0^{15} \left[200(10) + 2(100) - 120y\right] dy$$

$$= \left[2000y + 200y - 60y^2\right]_0^{15} = \boxed{19500}$$

2. ***Set up*** an integral for the volume of the solid under the paraboloid $z = x^2 + y^2$ and above the region bounded by $y = x^2$ and $x = y^2$.



$$\int_0^1 \int_{x^2}^{\sqrt{x}} x^2 + y^2 \, dy\, dx$$

Very nice!

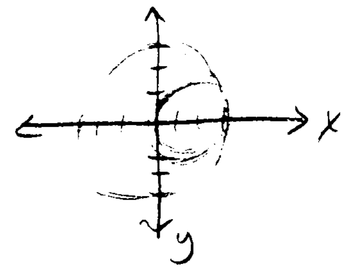
$z = \sqrt{9-x^2-y^2}$

3. *Set up* an integral to represent the volume between the surface $x^2 + y^2 + z^2 = 9$ and the xy plane within the circle $r = 2\cos\theta$.

$$2\int_0^{\pi/2}\int_0^{2\cos\theta}\left(\sqrt{9-x^2-y^2}\right) R\,dR\,d\theta$$

$$2\int_0^{\pi/2}\int_0^{r=2\cos\theta} R\sqrt{9-R^2}\,dR\,d\theta$$

TOP VIEW



Excellent

4. *Set up* integrals to find the center of mass of the trapezoid with density function $\rho(x,y)=2x+y$ and having vertices at (0,0), (1,0), (1,2), and (0,4).

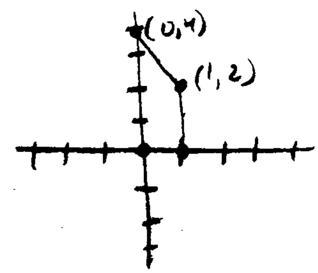
$$m = \int_0^1\int_0^{-2x+4} 2x+y\,dy\,dx$$

$$M_y = \int_0^1\int_0^{-2x+4} x(2x+y)\,dy\,dx$$

$$M_x = \int_0^1\int_0^{-2x+4} y(2x+y)\,dy\,dx$$

$$\bar{x} = \frac{M_y}{m} \quad \bar{y} = \frac{M_x}{m}$$

Great



$$\frac{y_2-y_1}{x_2-x_1} = \frac{2-4}{1-0} = \frac{-2}{1}$$

$$y = -2x+4$$

5. Find the Jacobian of the transformation x = u sin v, y = u cos v.

$$\frac{\partial x}{\partial u} = \sin v \qquad \frac{\partial y}{\partial u} = \cos v$$

$$\frac{\partial x}{\partial v} = u\cos v \qquad \frac{\partial y}{\partial v} = -u\sin v$$

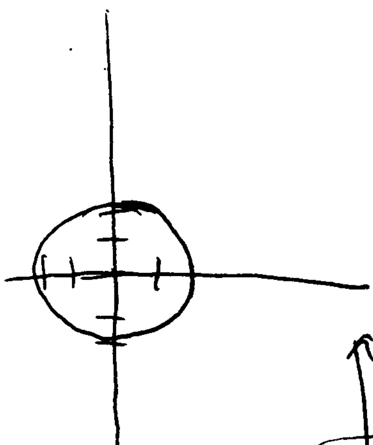
$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \sin v & \cos v \\ u\cos v & -u\sin v \end{vmatrix}$$

$$= -u\sin^2 v - u\cos^2 v = -u\cdot(\cancel{\sin^2 v + \cos^2 v})^{1} = \boxed{-u}$$

Well done

6. *Set up* an integral for the surface area of the portion of $z=\sqrt{x^2+y^2}$ below the plane z=2.

top view

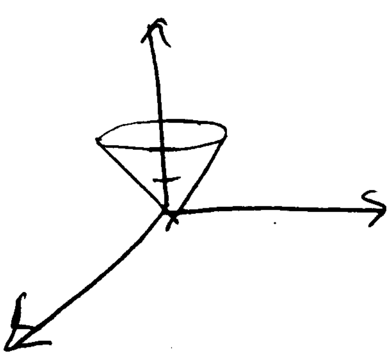


$$z = \sqrt{x^2+y^2} = (x^2+y^2)^{1/2}$$

$$f_x = \frac{1}{2}(x^2+y^2)^{-1/2}\cdot 2x = \frac{x}{\sqrt{x^2+y^2}}$$

$$f_y = \frac{1}{2}(x^2+y^2)^{-1/2}\cdot 2y = \frac{y}{\sqrt{x^2+y^2}}$$

$$\sqrt{1 + \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}} = \sqrt{2}$$



$$\boxed{\int_0^{2\pi}\int_0^{2} \sqrt{2}\, r\,dr\,d\theta}$$

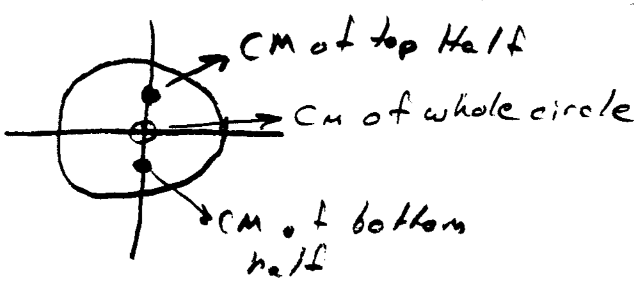
Nice

7. Bobbie-Sue is a calculus student at O.S.U. who's having a little trouble with multiple integrals. She says "Golly, I sure do like these here multiple integrals. The only thing is, I can't seem to get it into my head when you get to just do, like, half of something and double it, or maybe just the one quadrilateral and then take it times four. That whole symmetrical thing, you know?"

Give as clear an explanation as you can, in terms Bobbie-Sue can understand, of when it is and isn't possible to use symmetry.

Symmetry is a very usefal tool, if you know when to use it. When working with clearly symmetric objects like circles or spheres, symmetry can be used but you still have to be careful, Say You want the area of a circle, $x^2+y^2=1$ you could use symmetry,

since its the same in both halves. but if you wanted to find the center of mass, you could Not use symmetry, because each half has a different C.M. relative to the origin.



Excellent

8. *Set up* integrals in spherical coordinates for the z coordinate of the center of mass of the first-octant portion of the sphere $x^2 + y^2 + z^2 = 9$ (with uniform density).

density = k

$\bar{z} = \dfrac{M_{xy}}{m}$

$M_{xy} = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^{3} \square\, k\,\rho^2 \sin\phi \, d\rho\, d\theta\, d\phi \longrightarrow \boxed{M_{xy} = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^{3} \rho\cos\phi\, k\,\rho^2 \sin\phi\, d\rho\, d\theta\, d\phi}$

$z = \rho\cos\phi$

$0 \le \theta \le \pi/2$

$0 \le \rho \le 3$

$0 \le \phi \le \pi/2$

(limits)

$x^2 + y^2 + z^2 = \rho^2$

Very nice!

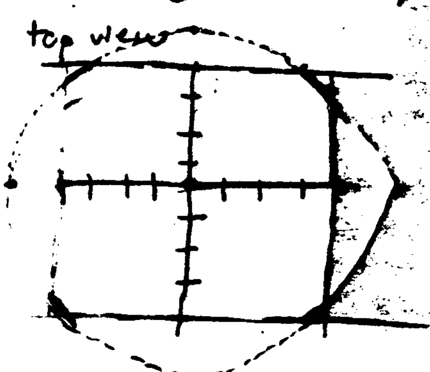
$\boxed{m = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^{3} k\,\rho^2 \sin\phi\, d\rho\, d\theta\, d\phi}$

9. Chris the mathematician plans to go into business selling mathematically designed paperweights. The paperweights will be shaped like the solid bounded by $z = 25 - x^2 - y^2$, but then sawed off at the planes $x = 4$, $x = -4$, $y = 4$, and $y = -4$. *Set up* an integral or integrals which will give the volume of one of the paperweights.

$z = 25 - x^2 - y^2 \quad x = 4 \quad x = -4 \quad y = 4 \quad y = -4$

$x^2 + y^2 = 25 \qquad y = \sqrt{25 - x^2}$

~~bound by~~ $\cancel{4 = \sqrt{25 - x^2}}$



$$4 \int_{x=0}^{3} \int_{y=0}^{4} \int_{0}^{25 - x^2 - y^2} dz\, dy\, dx \; +$$

$$4 \int_{3}^{4} \int_{0}^{\sqrt{25 - x^2}} \int_{0}^{25 - x^2 - y^2} dz\, dy\, dx$$

Symmetrical!

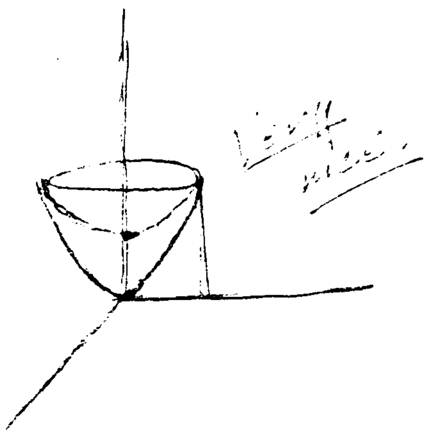
10

Beautifully done!

10. President Boren has been informed that there is a spot on the OU campus which lies more than a block from the nearest fountain (nobody can figure out where such a spot might be, but the threat is still being taken very seriously), and has issued a call for immediate action. The plan is to build a fountain with a crimson granite bowl shaped like the solid bounded by the paraboloids $z = 2x^2 + 2y^2$ and $z = x^2 + y^2 + 1$ (with all axes measured in meters) surrounded by a large pool, so that the water will cascade over the lip of the bowl down into the pool below in a manner intended to symbolize the way OU's football success leads to academic excellence. Find the volume of granite needed to form the bowl.

$z = 2x^2 + 2y^2$ $\qquad$ $z = x^2 + y^2 + 1$

$z = 2r^2$ $\qquad$ $z = r^2 + 1$



$$\int_0^{2\pi}\int_0^1\int_{2r^2}^{r^2+1} 1\, r\, dz\, dr\, d\theta$$

$$\int_0^{2\pi}\int_0^1 (r^2 + 1 - 2r^2)\, r\, dr\, d\theta$$

$$\int_0^{2\pi}\int_0^1 r - r^3\, dr\, d\theta$$

$$\int_0^{2\pi} \frac{r^2}{2} - \frac{r^4}{4}\Big|_0^1\, d\theta$$

$$\int_0^{2\pi} \frac{1}{4}\, d\theta = \frac{\theta}{4}\Big|_0^{2\pi} = \frac{\pi}{2}\ m^3$$