Exam 1 Calc IV (Math2443-002) 6/14/2002

Each problem is worth 10 points. Show all work for full credit. Please circle all answers and keep your work as legible as possible. Careful of that third step, it's a doozy.

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1. Find $\lim_{(x,y)\to(0,0)} \frac{3xy}{x^2+xy+y^2}$ or show that the limit does not exist.

If $x=0 \Rightarrow \lim_{(0,y)\to(0,0)} \frac{3xy}{x^2+y^2+y^2} \Rightarrow \lim_{y\to 0} \frac{0}{2y^2} = 0$

If $y=0 \Rightarrow \lim_{(x,0)\to(0,0)} \frac{3xy}{x^2+y^2+y^2} \Rightarrow \lim_{x\to 0} \frac{0}{2y^2} = 0$

If $x=y \Rightarrow \lim_{(y,y)\to(0,0)} \frac{3yy}{y^2+y^2+y^2} \Rightarrow \lim_{y\to 0} \frac{3y^2}{3y^2} = 1$

Limit DNE b/c when you approach from multiple directions you get different answers.

Great

W

2. Find the directional derivative of f(x,y) = e^{x-y} at the point (1,2) in the direction of the vector v = <-3,4>.

First, let's change the vector <-3,4> to the unit vector <a,b>

$$u = \left\langle \frac{-3}{\sqrt{(-3)^2+(4)^2}}, \frac{4}{\sqrt{(-3)^2+(4)^2}} \right\rangle = \left\langle \frac{-3}{5}, \frac{4}{5} \right\rangle.$$

Next, let's find the partials with respect to both x and y.

$$f(x,y) = e^{x-y}$$

$$f_x(x,y) = e^{x-y}$$

$$f_y(x,y) = -e^{x-y}$$

Then we plug in our point (1,2) into our partials

$$f_x(1,2) = e^{1-2} = e^{-1}$$

$$f_y(1,2) = -e^{1-2} = -e^{-1}$$

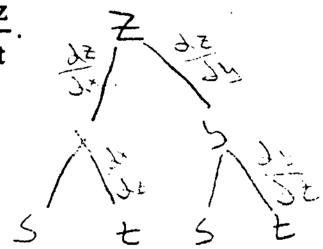
Great Job

Then we can find our directional derivative.

$$\vec{D}_u f(x,y) = a \cdot f_x(x,y) + b \cdot f_y(x,y)$$ where a and b is the unit vector <a,b>.

$$\vec{D}_u f(x,y) = \left(-\frac{3}{5} \cdot (e^{-1}) + \frac{4}{5}(-e^{-1}) \right) = -\frac{7}{5}e^{-1}.$$

3. If we have z = f(x,y,z) where x = x(s,t), and y = y(s,t), write out the appropriate chain rule for $\frac{\partial z}{\partial t}$.



$$\boxed{\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial t}\right) + \frac{\partial z}{\partial y}\left(\frac{\partial y}{\partial t}\right)}$$

Yes

4. Find an equation for the plane tangent to $f(x,y) = x^2y - y^2x + 7x$ at the point $(-3,1)$.

$f(x,y) = x^2y - y^2x + 7x$

Find partial der.

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$f_x(x,y) = 2xy - y^2 + 7$ $\quad f_x(-3,1) = 2(-3)(1) - (1)^2 + 7 = 0$

$f_y(x,y) = x^2 - 2yx$ $\quad f_y(-3,1) = (-3)^2 - 2(1)(-3) = 15$

General Equation

$z - z_0 = f_x(x - x_0) + f_y(y - y_0)$

Find z_0:

$z_0 = f(-3,1) = (-3)^2(1) - (1)^2(-3) + 7(-3) = 9 + 3 - 21 = -9$

$z - (-9) = 0(x+3) + 15(y-1)$

$z + 9 = 15(y-1)$

$z + 9 = 15y - 15$

$\boxed{z = 15y - 24}$

Well done

5. Find the maximum rate of change of $f(x,y) = xe^{-y} + 3y$ at the point (1,0) and the direction in which it occurs.

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Max rate of change: $|\nabla f|$ → mag. of gradient

direction it occurs is $\nabla f = \langle a, b \rangle$

$\nabla f = \langle f_x, f_y \rangle$

$f_x = e^{-y}$

$f_y = -xe^{-y} + 3$

@ (1,0) $f_x = e^{-0} = e^0 = \underline{1}$

$f_y = -1(e^0) + 3 = \underline{2}$

direction → $\underline{\nabla f = \langle 1, 2 \rangle}$

$|\nabla f| = \sqrt{1^2 + 2^2} = \underline{\underline{\sqrt{5}}}$ max rate of change

Great Job

6. Find the critical points of the function $f(x,y) = x^2 + 3y^2 + x^2y + 5$ and classify them as local maxima, minima, or saddle points.

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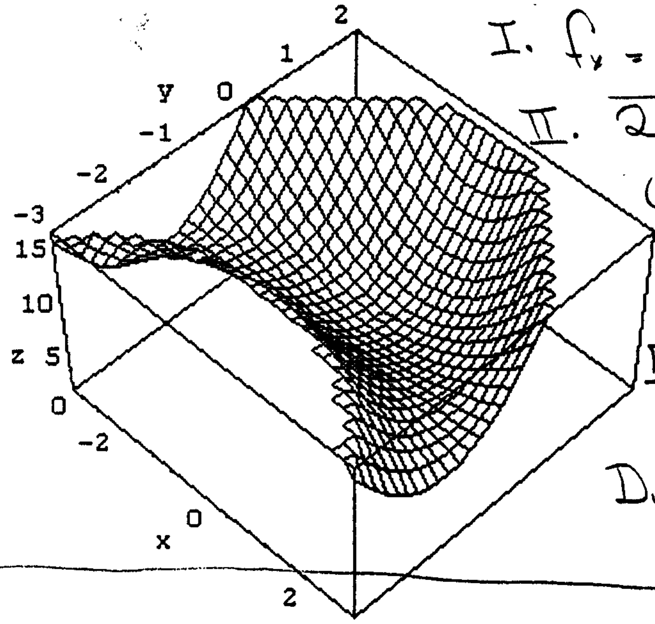

I. $f_x = 2x + 2xy$ $\quad f_y = 6y + x^2$

II. $2x + 2xy = 0 = 2x(1+y)$

$6y + x^2$ $\quad x = 0$ or $y = -$

$\underbrace{(0,0),\ (+\sqrt{6},-1),\ (-\sqrt{6},-1)}_{\text{C. pt.}}$

III $F_{xx} = 2 + 2y$ $\quad F_{yy} = 6$ $\quad F_{xy} =$ F_{yx}

$D_{(x,y)} = F_{xx}(F_{yy}) - (F_{xy})^2$

1. $D(0,0) = 2(6) - 0^2 = 12 > 0$ and $F_{xx} > 0$ so, min
2. $D(\sqrt{6},-1) = 0(6) - (2\sqrt{6})^2 = -24 < 0$ so saddle point
3. $D(-\sqrt{6},-1) = 0(6) - (2(-\sqrt{6}))^2 = -24 < 0$ so saddle point

Well done!

7. Jebediah is a calculus student as O.S.U. who's having some trouble with directional derivatives. Jeb says "Gosh darn it, I think I just spread a whole load of manure all over my calc test again. There was this one question about those level curve doo-hickys, and said some stuff about how all them level curves of this function was straight lines. I figured the only way that could happen was if the function was just a plane, so I did it like it was a plane. But then some girls was talking after the test and they was saying something totally different. Jeez, I hope I don't fail and have to retake that dang class a third time!"

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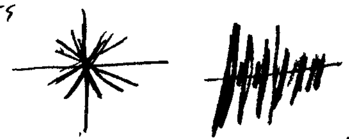
Help Jeb out by explaining (in a way he can understand!) what you can say about a surface if you know that all of its level curves are straight lines.

Well, First lets take a look at the level curves. If you look down at a graph from above (meaning keep the z-values constant) you should be seeing the cross-section that is being projected at a certain value of z (or a level curve). You should see a straight line. Now change the value of z. You should still see a straight line, however, it can be moved on the xy coordinate plane right, left, and/or spinning. If you keep doing this, you should be able to visualize the 3D shape. If the level curves keep moving in a right or left motaon at a constant rate, or if they all stay in the same position, yes, you do have a plane, So, Jeb is not completly wrong, but not completly right. It can spin or move ~~s~~back and forth so they don't form a plane but can form a spiral (like rotinni pasta or screw) or a wave.

EGs



SPIRAL   WAVE (HIGHER CROSS-SECTIONS ARE LONGER)

Excellent

8. Show that the equation of the tangent plane to the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ at the point (x_0,y_0,z_0) can be written as $\frac{xx_0}{a^2}+\frac{yy_0}{b^2}+\frac{zz_0}{c^2}=1$.

$$f_x(x,y,z) = \frac{2x}{a^2} + \frac{2z}{c^2}\frac{\partial z}{\partial x} = 0 \qquad \frac{\frac{2x}{a^2}}{\frac{2z}{-c^2}} = \frac{-\frac{2z}{c^2}\frac{\partial z}{\partial x}}{-\frac{2z}{c^2}} \qquad \frac{\partial z}{\partial x} = -\frac{2xc^2}{2za^2} = \frac{-2}{2z_0}$$

$$f_y(x,y,z) = \frac{2y}{b^2} + \frac{2z}{c^2}\frac{\partial z}{\partial y} = 0 \qquad \frac{\frac{2y}{b^2}}{-\frac{2z}{c^2}} = \frac{-\frac{2z}{c^2}\frac{\partial z}{\partial y}}{-\frac{2z}{c^2}} \qquad \frac{\partial z}{\partial y} = -\frac{2yc^2}{2zb^2} = \frac{-}{2z}$$

$$z - z_0 = -\frac{2x_0c^2}{2z_0a^2}(x-x_0) - \frac{2y_0c^2}{2z_0b^2}(y-y_0)$$

$$z - z_0 = -\frac{\cancel{2}xx_0c^2}{\cancel{2}z_0a^2} + \frac{\cancel{2}x_0^2c^2}{\cancel{2}z_0a^2} - \frac{\cancel{2}yy_0c^2}{\cancel{2}z_0b^2} + \frac{\cancel{2}y_0^2c^2}{\cancel{2}z_0b^2}$$

$$\frac{z_0z}{c^2} - \frac{z_0^2}{c^2} = -\frac{xx_0}{a^2} + \frac{x_0^2}{a^2} - \frac{yy_0}{b^2} + \frac{y_0^2}{b^2}$$

$$\frac{zz_0}{c^2} + \frac{xx_0}{a^2} + \frac{yy_0}{b^2} = \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} + \frac{z_0^2}{c^2}$$

$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} + \frac{zz_0}{c^2} = 1 \leftarrow$ since $\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} + \frac{z_0^2}{c^2}$ must fulfill the original equation.

Very nicely done!

9. Mike the mountain goat is standing on a pleasant mountaintop shaped exactly like the surface $z = 5000 - x^2 - y^2$. Now Mike especially likes points where the slope in the direction where it's steepest is exactly 2, so he's looking around himself wondering if there are any such points. Tell Mike where to find the points he'll like.

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$\frac{\partial z}{\partial x} = -2x$ $\qquad$ $\frac{\partial z}{\partial y} = -2y$

If Mike likes it when the slope is 2, that means the magnitude of the direction vector must be 2.

direction vector $\langle -2x, -2y \rangle$

Excellent

$\sqrt{(-2x)^2+(-2y)^2} = 2$

$4x^2+4y^2 = 4$

$x^2+y^2=1$

$z = 5000-x^2-y^2$

$z+x^2+y^2=5000$

$z+1=5000$

$z=4999$

$\therefore$ At height 4999, Mike can stand anywhere along $x^2+y^2=1$, and the slope will be 2.

10. Find a function of the form $p(x,y) = ax^2 + by^2 + cxy + dx + ey + f$ (**you** figure out the values of a, b, c, d, e, and f) which meets the following list of requirements:

$p(0,0) = -7$
$p_x(0,0) = 12$
$p_y(0,0) = -6$
$p_{xx}(0,0) = 6$
$p_{xy}(0,0) = 1$
$p_{yy}(0,0) = -4$

W

$p_x = 2ax + cy + d = 12$

$p_y = 2by + cx + e = -6$

$p_{xx} = 2a = 6$ $\quad 2a = 6 \quad a = 3$

$p_{xy} = c = 1$ $\quad c = 1$

$p_{yy} = 2b = -4$ $\quad 2b = -4 \quad b$

$a = 3$
$b = -2$
$c = 1$
$d = 12$
$e = -6$
$f = -7$

Well done

$2(3)(0) + 1(0) + d = 12$

$d = 12$

$2(-2)(0) + c(0) + e = -6$

$e = -6$

$3(0)^2 + b(0)^2 + 1(0)(0) + 12(0) + -6(0$

$f = -7$