Exam 2 Calc 1 7/7/2005

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. State and prove the Constant Rule for derivatives.

$\boxed{\text{If } f(x) = k \text{ for some constant } k, \text{ then } f'(x) = 0.}$

Proof: $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$= \lim_{h \to 0} \dfrac{k - k}{h}$

$= \lim_{h \to 0} \dfrac{0}{h}$

$= 0$ $\square$

2. Given the table of values at right, find the following:

x	f(x)	f′(x)	g(x)	g′(x)
0	5	-6	2	0
1	2	1	-3	5
2	7	3	-1	17
3	2	-4	0	2

a) F′(1), if F(x) = (f / g)(x).

$F'(x) = \dfrac{f'(1) \cdot g(1) - f(1) \cdot g'(1)}{[g(1)]^2}$

$F'(1) = \dfrac{1 \cdot -3 - 2 \cdot 5}{(-3)^2} = -\dfrac{13}{9}$

b) G′(2), if G(x) = $e^{(g(x))}$.

$G'(x) = e^{(g(x))} \cdot g'(x)$

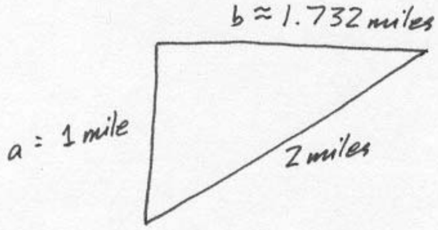
$G'(2) = e^{(g(2))} \cdot g'(2) = e^{-1} \cdot 17 = \dfrac{17}{e}$

c) H′(3), if H(x) = (f ∘ g)(x). $= f(g(x))$

$H'(x) = f'(g(x)) \cdot g'(x)$

$H'(3) = f'(g(3)) \cdot g'(3) = f'(0) \cdot 2 = -6 \cdot 2 = -12$

3. A plane flying horizontally at an altitude of 1 mile and a speed of 600 miles per hour passes directly over Jon's house (which is just a few miles from the airport). Find the rate at which the distance from the plane to the house is increasing when it is 2 miles away from the house.



First I'll find the length of the other leg of this triangle:

$$(1)^2 + b^2 = (2)^2$$
$$b^2 = 4 - 1$$
$$b = \sqrt{3}$$ (choosing the positive root)

$$\frac{da}{dt} = 0$$
$$\frac{db}{dt} = 600 \text{ mi/hr}$$
$$\frac{dc}{dt} = ?$$

Now I need a relation among the rates of change, so I differentiate:

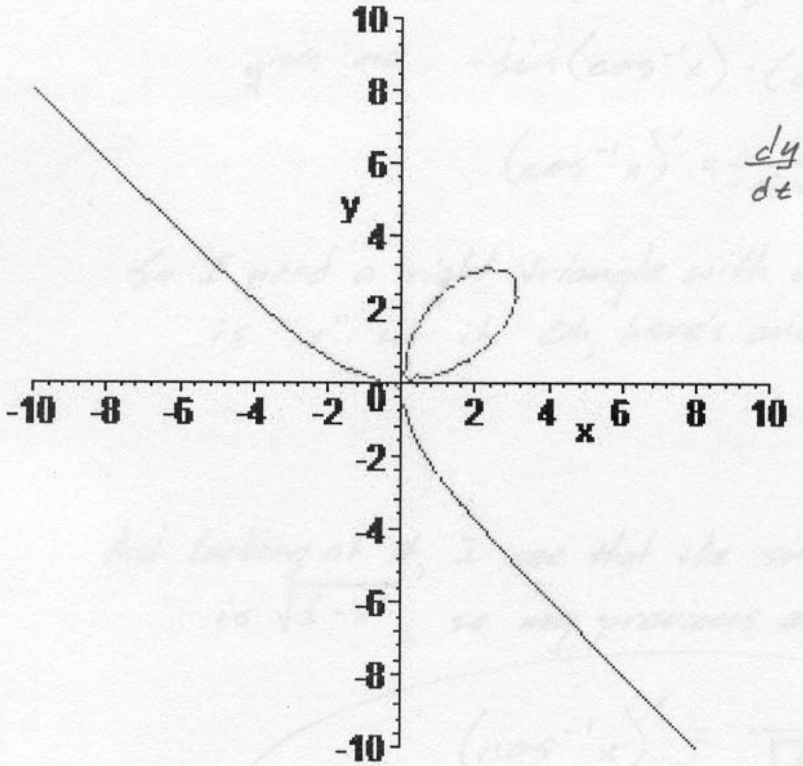
$$a^2 + b^2 = c^2$$
$$2a \cdot \frac{da}{dt} + 2b \cdot \frac{db}{dt} = 2c \cdot \frac{dc}{dt}$$

Then plug in what I know:

$$2(1)(0) + 2(1.732)(600) = 2(2)\frac{dc}{dt}$$
$$2078.4 = 4\frac{dc}{dt}$$
$$\frac{dc}{dt} \approx 520 \text{ miles per hour}$$

4. The curve with equation $x^3 + y^3 = 6xy$ is called the folium of Descartes. Find the equation of its tangent line at the point (3,3).

Differentiate Implicitly:



$$3x^2 + 3y^2 \cdot \frac{dy}{dt} = 6y + 6x \cdot \frac{dy}{dt}$$

$$\frac{dy}{dt}(3y^2 - 6x) = 6y - 3x^2$$

$$\frac{dy}{dt} = \frac{6y - 3x^2}{3y^2 - 6x}$$

So at the point (3, 3):

$$\frac{dy}{dt} = \frac{6(3) - 3(3)^2}{3(3)^2 - 6(3)}$$

$$= \frac{18 - 27}{27 - 18}$$

$$= \frac{-9}{9}$$

$$= -1$$

So we want a line with a slope of -1 through (3,3):

$$y - 3 = -1(x - 3)$$

$$y = -x + 6$$

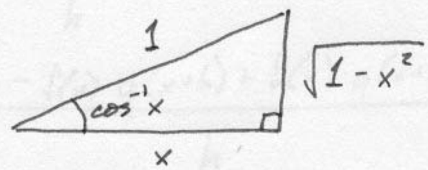
5. Show that $\left(\cos^{-1} x\right)' = \dfrac{-1}{\sqrt{1-x^2}}$.

Well, I know $\cos\left(\cos^{-1} x\right) = x$, so differentiating

gives me $-\sin\left(\cos^{-1}x\right)\cdot\left(\cos^{-1}x\right)' = 1$, or

$$\left(\cos^{-1}x\right)' = \frac{1}{-\sin\left(\cos^{-1}x\right)}$$

So I need a right triangle with an angle whose cosine is "x" in it. Oh, here's one!



And looking at it, I see that the sine of that angle is $\sqrt{1-x^2}$, so my previous answer can be simplified:

$$\left(\cos^{-1}x\right)' = \frac{-1}{\sqrt{1-x^2}}$$

6. State and prove the Product Rule for derivatives.

If f and g are differentiable functions, then

$$(f\cdot g)' = f'\cdot g + f\cdot g'.$$

Proof: Well, $\displaystyle (f\cdot g)'(x) = \lim_{h\to 0} \frac{(f\cdot g)(x+h) - (f\cdot g)(x)}{h}$

$$= \lim_{h\to 0} \frac{f(x+h)\cdot g(x+h) - f(x)\cdot g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)\cdot g(x+h) - f(x)\cdot g(x+h) + f(x)\cdot g(x+h) - f(x)\cdot g(x)}{h}$$

$$= \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\cdot g(x+h)\right] + \lim_{h\to 0}\left[f(x)\cdot \frac{g(x+h)-g(x)}{h}\right]$$

$$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \cdot \lim_{h\to 0} g(x+h) + \lim_{h\to 0} f(x)\cdot \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$$

$$= f'(x)\cdot g(x) + f(x)\cdot g'(x). \quad \square$$

7. If $f(x) = x^{\tan x}$, find $f'(x)$.

Logarithmic differentiation!

$y = x^{\tan x}$

$\ln y = \ln x^{\tan x}$

$\ln y = (\tan x) \cdot \ln x$ Now take derivatives.

$\frac{1}{y} \cdot y' = \sec^2 x \cdot \ln x + \tan x \cdot \frac{1}{x}$ (Product Rule on the right)

$y' = \left(\sec^2 x \cdot \ln x + \frac{\tan x}{x}\right) \cdot y$

$y' = \left(\sec^2 x \cdot \ln x + \frac{\tan x}{x}\right) \cdot x^{\tan x}$

8. Biff is a calculus student at Enormous State University, and he's having some trouble. Biff says "Dang, this calculus stuff is kicking my ass. I was gonna be a math major, but now I have no idea what's going on and I might switch to philosophy or something. But I still want to get this stuff. So here's what I'm trying to figure out: The T.A. said we don't really know any of this stuff unless we know how it comes from the definition of the derivative, right? And I'm good with that, because I like knowing why things are and stuff. So we learned how you prove the Product Rule and Quotient Rule and stuff from the definition, but not the Chain Rule. So does that mean we don't really know the Chain Rule agrees with the definition of the derivative?"

Help Biff out by explaining why we might have good reason to believe that the Chain Rule agrees with the definition fo the derivative, even without a complete proof.

Hey Biff,

So the thing is, actually proving the Chain Rule is too hard for Calc 1, but still there are lots of clues to suggest it. In our class we worked out derivatives by the Product Rule for things like $(e^{2x})' = (e^x \cdot e^x)' = e^x \cdot e^x + e^x \cdot e^x = 2e^{2x}$, $(e^{3x})' = (e^{2x} \cdot e^x)' = 2e^{2x} \cdot e^x + e^{2x} \cdot e^x = 3e^{3x}$. You can do some compositions with trig functions too, by using trig identities to get $[\cos(2x)]' = -2\sin(2x)$ and $[\sin(2x)]' = 2\cos(2x)$.

So even if we can't prove the Chain Rule, we can do lots of derivatives by other ways that we <u>have</u> proved, and see that the results are the same as what you get when you use the actual Chain Rule. That's not as good as an actual proof, but generalizing a pattern is at least a good start.

9. If $h(x) = f(x)g(x)$, where f and g are functions whose first, second, and third derivatives exist,

a) Show that $h''(x) = f''g + 2f'g' + g''$.

$$h'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

$$h''(x) = f'' \cdot g + f' \cdot g' + f \cdot g'' + f' \cdot g'$$

$$h''(x) = f''g + 2f'g' + g''f$$

Great

b) Derive a formula for h''' similar to the one in part a.

$$h''(x) = f''g + 2f'g' + g''f$$

$$h'''(x) = f'''g + f''g' + 2(f''g' + f'g'') + g'''f + g''f'$$

$$f'''g + f''g' + 2f''g' + 2f'g'' + g'''f + g''f'$$

$$h'''(x) = f'''g + 3f''g' + 3f'g'' + g'''f$$

Excellent!

10. The equation

$$y'' + 5y' - 6y = 0$$

is called a **differential equation** because it involves an unknown function y and its derivatives y' and y''. Find a constant r such that the function

$$y = e^{rx}$$

satisfies this equation.[1] [Hint: Start by finding y' and y'', then substitute them into the differential equation and see what you can do.]

Well, $y = e^{r \cdot x}$, so

$y' = r \cdot e^{rx}$ and

$y'' = r^2 \cdot e^{rx}$.

So if $y'' + 5y' - 6y = 0$,

$$(r^2 \cdot e^{rx}) + 5(r \cdot e^{rx}) - 6(e^{rx}) = 0,$$

or $\quad e^{rx}(r^2 + 5r - 6) = 0$

or $\quad e^{rx}(r + 6)(r - 1) = 0$

and since no values of r or x will make e^{rx} equal zero, it must be that one of the other factors is zero, so either $r = -6$ or $r = 1$.