Exam 1　　Calc 1　　6/22/2006

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. State the definition of the derivative of the function f(x) at the point $x = a$.

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Use the graph of f(x) at the bottom of the page for problems 2 and 3:

2

a) What is f(-1)?　　$f(-1) = 2$

b) What is $\lim_{x \to -1} f(x)$?　　$\lim_{x \to -1} f(x) = 1$

Great!

c) What is $\lim_{x \to 2^+} f(x)$?　　$\lim_{x \to 2^+} f(x) = 2$

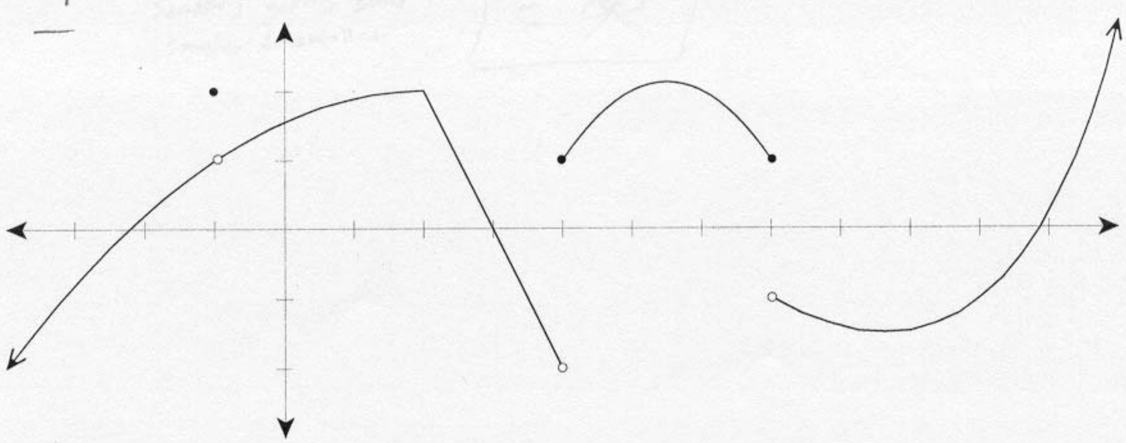
3. a) For which value(s) of x is f(x) not continuous? Why?

-1, $\lim_{x \to -1} f(x) \neq f(-1)$　　4, $\lim_{x \to 4^-} f(x) \neq \lim_{x \to 4^+} f(x)$　　7, $\lim_{x \to 7^-} f(x) \neq \lim_{x \to 7^+} f(x)$

b) For which value(s) of x is f(x) not differentiable? Why?

-1, 4, 7 Not continuous　　2, corner

Great



4. Estimate $\lim_{x\to 1}\dfrac{\sqrt[3]{x}-1}{\sqrt{x}-1}$ numerically.

x	$f(x)$
.9	.67250
.99	.66722
.999	.66672
.9999	.66667
.99999	.66667
1.00001	.66666
1.0001	.66666
1.001	.66661
1.01	.66611
1.1	.66136

Excellent

$$\lim_{x\to 1}\frac{\sqrt[3]{x}-1}{\sqrt{x}-1}=\frac{2}{3}$$

5. Evaluate $\lim_{x\to\infty}\dfrac{\sqrt{9x^4-x^2}}{25-x^2}$ exactly.

Multiply top + bottom by $\frac{1}{x^2}$:

$$=\lim_{x\to\infty}\frac{\sqrt{\frac{9x^4}{x^4}-\frac{x^2}{x^4}}}{\frac{25}{x^2}-\frac{x^2}{x^2}}=\lim_{x\to\infty}\frac{\sqrt{9-\frac{1}{x^2}}}{\frac{25}{x^2}-1}=\frac{\sqrt{9}}{-1}=-3$$

6. If f(x) = $\sqrt{3x+1}$, use the definition of the derivative to find f$'$(2).

$$f'(2) = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{3(2+h)+1} - \sqrt{7}}{h} \cdot \frac{\sqrt{3(2+h)+1} + \sqrt{7}}{\sqrt{3(2+h)+1} + \sqrt{7}}$$

$$= \lim_{h\to 0} \frac{3(2+h)+1 - 7}{h\left(\sqrt{3(2+h)+1} + \sqrt{7}\right)}$$

$$= \lim_{h\to 0} \frac{\cancel{6} + 3h \cancel{+1} \cancel{-7}}{h\left(\sqrt{6+3h+1} + \sqrt{7}\right)}$$

$$= \lim_{h\to 0} \frac{3\cancel{h}}{\cancel{h}\left(\sqrt{3h+7} + \sqrt{7}\right)}$$

$$= \lim_{h\to 0} \frac{3}{\sqrt{3h+7} + \sqrt{7}}$$

$$= \lim_{h\to 0} \frac{3}{\sqrt{3(0)+7} + \sqrt{7}}$$

$$= \frac{3}{\sqrt{7} + \sqrt{7}}$$

$$\boxed{= \frac{3}{2\sqrt{7}}}$$

Nicely done!

7. Evaluate the limit $\lim_{x\to 3}\dfrac{x^2}{2x-1}$ and justify each step by indicating the appropriate limit law(s) from the list below.

$$= \frac{\lim_{x\to 3} x^2}{\lim_{x\to 3}(2x-1)} = \frac{\left(\lim_{x\to 3} x\right)^2}{\lim_{x\to 3} 2x - \lim_{x\to 3} 1} = \frac{(3)^2}{2\lim_{x\to 3} x - 1} = \frac{9}{2(3)-1} = \frac{9}{5}$$

Quotient Rule

Power Rule

Difference Rule

Rule X

Constant Mult. Rule

Constant Rule

Rule X

8. Bunny is a calculus student at Enormous State University, and she's having some trouble. Bunny says "Ohmygod, we had this test, and it was so messed up. In high school math was always just working out problems, but now there are these problems where, like, they want you to say *why* something is what it is. I thought that was philosophy or something. But so there was this one question on our test about whether this equation $e^x = 2 - x$ had a solution, and when the grad student who runs our class handed the test back, he was saying all this stuff about how we were supposed to give some important reason to explain it. I don't know how to explain it, I just tried lots of things to solve it and couldn't!"

Explain to Bunny, as clearly as possible, either how she might assure that there is a solution to this equation, or how she could show that no solution exists.

Actually Bunny, that looks a lot like a homework problem from our book. Maybe you could have seen it before the exam?

The key is you don't have to find a solution, just decide if there is one. The Intermediate Value Theorem helps with that. Use the function $f(x) = 2 - x - e^x$, since any root of that function will be a solution to the original equation (think about it). Since $f(0) = 1$ but $f(1) =$ (negative something), and f is continuous, there has to be an input between 0 and 1 that produces an output of 0, because 0 is between 1 and any negative number.

9. Let f(x) = $mx + b$, where m and b are constants. Use the definition of the derivative to find f′(a).

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h\to 0} \frac{(m(a+h)+b) - (ma+b)}{h}$$

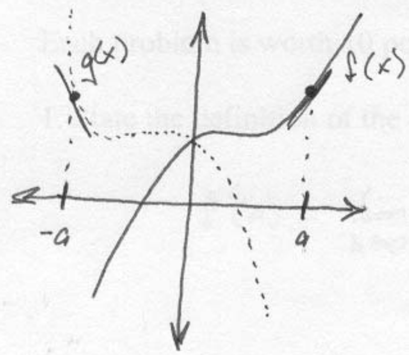
$$= \lim_{h\to 0} \frac{\cancel{ma} + mh \cancel{+b} - \cancel{ma} \cancel{-b}}{h}$$

$$= \lim_{h\to 0} \frac{m\cancel{h}}{\cancel{h}}$$

$$\boxed{= m}$$

Well done

10. Suppose that g(x) = f(-x). What connection would you expect between g′ and f′, and why?



Graphically, g and f are mirror images across the y-axis. That means if you look where $x = a$, f and g might not have any connection, but $g(-a)$ is just like $f(a)$ except backwards, so having the negative of $f(a)$'s slope. So I'd say $g'(a) = -f'(-a)$.