Exam 2 Calc 1 6/6/2006

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. State and prove the Constant Rule for derivatives.

If $f(x) = C$ with some constant C, then $f'(x) = 0$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad \lim_{h \to 0} f(x+h) = f(x)$$

$$= \lim_{h \to 0} \frac{C - C}{h}$$

$$\boxed{= 0}$$

Good

2. Find an equation for the line tangent to $f(x) = x + \ln(\sin x)$ at the point where $x = \pi/2$.

$$f'(x) = 1 + \frac{1}{\sin x} \cdot \cos x$$

$$f'(x) = 1 + \cot x$$

$$f'\left(\tfrac{\pi}{2}\right) = 1 + \cot\left(\tfrac{\pi}{2}\right)$$

$$= 1 + 0$$

$$= 1$$

$$f\left(\tfrac{\pi}{2}\right) = \tfrac{\pi}{2} + \ln\left(\sin\tfrac{\pi}{2}\right)$$

$$= \tfrac{\pi}{2} + \ln 1$$

$$= \tfrac{\pi}{2}$$

Great

$$y - \tfrac{\pi}{2} = x - \tfrac{\pi}{2}$$

$$\boxed{y = x}$$

3. Show that the derivative of cot x is $-\csc^2 x$.

$$(\cot x)' = \left(\frac{\cos x}{\sin x}\right)'$$

$$= \frac{-\sin x \cdot \sin x - \cos x \cdot \cos x}{\sin^2 x}$$

$$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}$$

$$= \frac{-1(\sin^2 x + \cos^2 x)}{\sin^2 x}$$

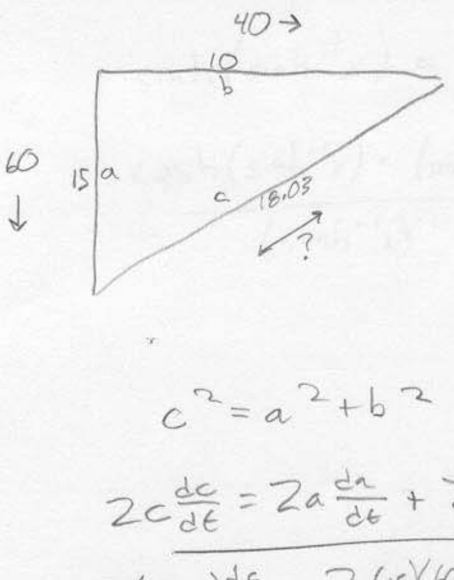
$$= \frac{-1(1)}{\sin^2 x}$$

$$= \frac{-1}{\sin^2 x}$$

$$\boxed{= -\csc^2 x}$$

Good

4. Following a bank robbery, two getaway cars leave a bank at exactly noon, one heading East at 40 miles per hour and the other heading South at 60 miles per hour. Fifteen minutes later the tracking devices hidden in the cash stolen from the bank activate. How fast is the distance between the two cars changing at that incredibly exciting instant?



$b = \frac{40}{4} = 10$

$a = \frac{60}{4} = 15$

$c = \sqrt{a^2 + b^2}$

$c = \sqrt{15^2 + 10^2}$

$c = 18.03$

$$c^2 = a^2 + b^2$$

$$2c\frac{dc}{dt} = 2a\frac{da}{dt} + 2b\frac{db}{dt}$$

$$2(18.03)\frac{dc}{dt} = 2(15)(60) + 2(10)(40)$$

$$\frac{dc}{dt} = \frac{2600}{36.06}$$

$$\boxed{\frac{dc}{dt} = 72 \ \text{mi/hr}}$$

Excellent

5. Find y' for the curve $x^3 - 3xy + y^3 = 1$.

Implicitly:

$$3x^2 - 3(1\cdot y + x\cdot y') + 3y^2\cdot y' = 0$$

$$3x^2 - 3y - 3xy' + 3y^2\cdot y' = 0$$

$$y'(-3x + 3y^2) = 3y - 3x^2$$

$$y' = \frac{3y - 3x^2}{3y^2 - 3x}$$

6. Show that $\left(\sinh^{-1} x\right)' = \dfrac{1}{\sqrt{1+x^2}}$ [Feel free to use the identity $\cosh^2 x - \sinh^2 x = 1$].

$$\sinh\left(\sinh^{-1}(x)\right) = x$$

{doing derivative of both sides using chain rule on left}

$$\cosh\left(\sinh^{-1}(x)\right)\cdot\left[\sinh^{-1}(x)\right]' = 1$$

$$\left[\sinh^{-1}(x)\right]' = \frac{1}{\cosh(\sinh^{-1}(x))}$$

$$\sinh^{-1}(x) = \frac{1}{\sqrt{1+x^2}}$$

$$\cosh^2(x) - \sinh^2(x) = 1$$

$$\cosh^2(x) = 1 + \sinh^2(x)$$

$$\sqrt{\cosh^2(x)} = \sqrt{1+\sinh^2(x)}$$

$$\cosh(x) = \sqrt{1+\sinh^2(x)}$$

Nice Job.

7. State and prove the Quotient Rule for derivatives.

If $f(x)$ + $g(x)$ are differentiable and $g(x) \neq 0$, then

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

First prove reciprocal rule: $\left(\frac{1}{g(x)}\right)' = -\frac{g'(x)}{[g(x)]^2}$

$$\left(\frac{1}{g(x)}\right)' = \lim_{h\to 0} \frac{\frac{1}{g(x+h)} - \frac{1}{g(x)}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{g(x+h)}\cdot\frac{g(x)}{g(x)} - \frac{1}{g(x)}\cdot\frac{g(x+h)}{g(x+h)}}{h}$$

$$= \lim_{h\to 0} \frac{g(x) - g(x+h)}{g(x+h)g(x)} \cdot \frac{1}{h}$$

Well done

$$= \lim_{h\to 0} \frac{-1[g(x+h) - g(x)]}{h} \cdot \frac{1}{g(x+h)g(x)}$$

$$= -g'(x) \cdot \frac{1}{g(x)g(x)}$$

$$= \frac{-g'(x)}{[g(x)]^2}$$

Then $\left(\frac{f}{g}\right)'(x) = \left(f\cdot\frac{1}{g(x)}\right)' = f'(x)\cdot\frac{1}{g(x)} + f(x)\cdot\frac{-g'(x)}{[g(x)]^2}$

$$= f'(x)\cdot\frac{1}{g(x)}\cdot\frac{g(x)}{g(x)} - \frac{f(x)g'(x)}{[g(x)]^2}$$

$$= \frac{f'(x)g(x)}{[g(x)]^2} - \frac{f(x)g'(x)}{[g(x)]^2}$$

$$\boxed{= \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}}\ . \ \square$$

8. Biff is a calculus student at Enormous State University, and he's having some trouble. Biff says "Dude, I found this weird thing in our Calculus book. It says the derivative of ln $(4x)$ is just $1/x$, but that's screwed up because that's supposed to be the derivative of just regular ln x, right? So there can't be two different functions with the same derivative, can there? So I figure the book's wrong somehow. Do you think I'll be famous for finding it out?"

Help Biff out by explaining either where his mistake is, or why it's okay that these two functions have the same derivative.

Biff, you did both derivatives right. In fact, it didn't matter that the constant was a 4. It could be any fixed number and the same thing will happen:

$$y = \ln(a \cdot x)$$

$$y' = \frac{1}{a \cdot x} \cdot (a \cdot x)' \qquad \text{[by the Chain Rule]}$$

$$y' = \frac{1}{a \cdot x} \cdot a \qquad \text{[by Const. Mult. Rule]}$$

$$y' = \frac{1}{x}$$

And you shouldn't be surprised by this case of two functions having the same derivative, because really it happens all the time. Any two constant functions have the same derivative, for instance, and $f(x) = x^2 + 3$ and $g(x) = x^2 - 2$ both have the same derivatives too. You might think $\ln(4x)$ isn't as much like $\ln x$ as $x^3 + 3$ is to $x^2 - 2$, but actually if you remember one of those abstruse identities, $\ln 4x = \ln 4 + \ln x$, and since $\ln 4$ is a constant this <u>has</u> to have the same derivative as $\ln x$.

9. Find all points on the circle $x^2 + y^2 = 25$ whose tangent lines pass through the point (7,1).

Differentiating implicitly:

$$2x + 2y \cdot y' = 0$$

$$2y \cdot y' = -2x$$

$$y' = -\frac{2x}{2y}$$

$$y' = -\frac{x}{y}.$$

So now we want a point (a, b) satisfying $x^2+y^2=25$ so that the line through it with slope $-\frac{x}{y}$ passes through $(7,1)$:

$$y-(b) = \left(-\frac{a}{b}\right)(x-(a)) \quad \text{must work with } (7,1), \text{ so}$$

$$1-b = -\frac{a}{b}(7-a)$$

$$b-b^2 = -7a+a^2$$

$$7a+b = a^2+b^2 \qquad \text{and } (a,b) \text{ was on } x^2+y^2=25, \text{ so}$$

$$7a+b = 25$$

$$b = 25-7a \qquad \text{again, this satisfies } x^2+y^2=25, \text{ so}$$

$$(a)^2 + (25-7a)^2 = 25$$

$$a^2 + 625 - 350a + 49a^2 = 25$$

$$50a^2 - 350a + 600 = 0$$

$$a^2 - 7a + 12 = 0$$

$$(a-3)(a-4) = 0$$

$$a = 3 \text{ or } a = 4 \qquad \text{and plugging these back into } b = 25-7a\text{:}$$

$$b = 25-7(3) \text{ or } b = 25-\cancel{7}(4)$$

$$b = 4 \text{ or } b = -3$$

So $(3,4)$ and $(4,-3)$ are the two such points.

10. If you know that f(x) is a function for which $f'(x) = [f(x)]^2$, what can you say about the derivative of $f^{-1}(x)$?

Well, I know $f(f^{-1}(x)) = x$, so differentiating,

$$f'(f^{-1}(x)) \cdot [f^{-1}(x)]' = 1$$

$$[f^{-1}(x)]' = \frac{1}{f'(f^{-1}(x))}$$

Now since $f'(x)$ is the same as $[f(x)]^2$,

$$[f^{-1}(x)]' = \frac{1}{[f(f^{-1}(x))]^2}$$

$$[f^{-1}(x)]' = \frac{1}{x^2}$$