BANANA

Exam 1 **Calculus 3** **9/18/2002**

Each problem is worth 10 points. Show adequate justification for full credit. Please circle all answers and keep your work as legible as possible. All warrantees void in case of improper use.

√ 1. Write the first four terms of the sequence $\left\{\frac{n-1}{n^2}\right\}$. $\frac{0}{1}$ $\frac{1}{4}$ $\frac{2}{9}$ $\frac{3}{16}$

$a_1 = 0$
$a_2 = 1/4$
$a_3 = 2/9$
$a_4 = 3/16$

10

√ 2. Write the first four partial sums of the series $\sum_{n=1}^{\infty} \frac{n-1}{n^2}$.

$S_1 = 0$
$S_2 = 1/4$
$S_3 = 1/4 + 2/9$
$S_4 = 1/4 + 2/9 + 3/16$

The partial sums just add each progressive a_n.

$S_n = a_n + a_{n-1} + a_{n-2} \ldots$

Excellent

3. Give an example of a series which is convergent but not absolutely convergent.

$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ conditionally convergent

$\sum_{n=1}^{\infty} \frac{1}{n} \Rightarrow$ divergent p-series w/p=1

⟹ alt. Ser. Test

(a) $\frac{1}{n+1} \overset{?}{\leq} \frac{1}{n}$

$\frac{(n+1)n}{n+1} \overset{?}{\leq} \frac{n(n+1)}{n}$

$n \leq n+1$

(b) $\lim_{n\to\infty} \frac{1}{n} = \frac{1}{\infty} = 0$

yes converges by alt. ser test

Great

4. Determine whether the series $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^5+n}}$ converges or diverges.

Try comp. ~~test~~

$n^5 + n > n^5$

$\sqrt{n^5+n} > \sqrt{n^5}$

$\frac{1}{\sqrt{n^5+n}} < \frac{1}{\sqrt{n^5}}$

$\frac{n}{\sqrt{n^5+n}} < \frac{n}{\sqrt{n^5}}$

$\frac{n}{n^{5/2}} = \frac{1}{n^{3/2}}$

$3/2 > 1$

$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges by the p-series test

$\therefore \sum_{n=1}^{\infty} \frac{n}{\sqrt{n^5+n}}$ converges by the comp. te

Excellent

converges

5. Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges or diverges.

Alt Series Test:

$\frac{1}{\sqrt{n}} > 0 \ \forall n$

$b_{n+1} \leq b_n$ $\qquad \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

$\frac{1}{\sqrt{n+1}} \leq \frac{1}{\sqrt{n}}$ the denom inc w/o bound as n in w/o bound, making the dif b/t zero + itself negligable

$\sqrt{n} \leq \sqrt{n+1}$

$n \leq n+1$

$0 \leq 1$

the seq is dec

Very nice!

$\frac{1}{\sqrt{n}}$ fulfills all of the requirements for the alt series test, $\therefore$ the series $\sum_{n=1} \frac{(-1)^n}{\sqrt{n}}$ converges

6. Determine whether the series $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$ converges or diverges.

Try Integral Test

$\int_1^{\infty} \frac{e^{1/x}}{x^2}$

$\lim_{b\to\infty} \int_1^{b} \frac{e^{1/x}}{x^2}$ let $u = \frac{1}{x}$, $du = \frac{-1}{x^2} \cdot dx$

Well done!

$\lim_{b\to\infty} -\int_1^{b} \frac{e^{1/x}}{1} \cdot \frac{-1}{x^2} \cdot dx$

$\lim_{b\to\infty} -\int_1^{b} e^{u} \cdot du$

$\lim_{b\to\infty} -e^{u}\Big]_1^{b}$

$\lim_{b\to\infty} -e^{\frac{1}{x}}\Big]_1^{b}$ → $\frac{1}{b}$ approaches 0

$\lim_{b\to\infty} -e^{\frac{1}{b}} + e$

$= -e^{0} + e$

$= e - 1$ converges

∴ since the integral converges the given series converges as well by the integral test

7. Write the third degree Taylor polynomial for the function f(x)=ln x centered at a=2.

n	$f^{(n)}$	$f^{(n)}2$
0	$\ln x$	$\ln 2$
1	$\frac{1}{x}$	$\frac{1}{2}$
2	$-\frac{1}{x^2}$	$-\frac{1}{4}$
3	$\frac{2}{x^3}$	$\frac{2}{8} = \frac{1}{4}$

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$f(x) = \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots$

$= \ln 2 + \frac{\frac{1}{2}}{1!}(x-2) + \frac{-\frac{1}{4}}{2!}(x-2)^2 + \frac{\frac{1}{4}}{3!}(x-2)^3$

$= \boxed{\ln 2 + \frac{1}{2}(x-2) - \frac{1}{8}(x-2)^2 + \frac{1}{24}(x-2)^3}$

x^{-1}

$-1x^{-2} = -\frac{1}{x^2}$

$-x^{-2} = -(-2)x^{-3}$

$\frac{2}{3}$

Great

8. Chaz is a calculus student at Enormous State University, and he's having trouble with series. Chaz says "Ya know, I used to be pretty good at math, but this series crap is just outta control. What's up with this thing where you do a bunch of work, and it turns out it's no good? Like with that ratio test thing, you know? You do it, and you get 1, and they say that means you have to try something else. It's like it's just a conspiracy or something, because I did it for that one over n squared series, and it was like a total waste of time, and the series converges anyway. So why the heck don't they just say if you get 1 from the ratio test, it's gonna converge?"

Explain to Chaz, in terms he can understand, whether a 1 from the ratio test means that a series converges or not.

A 1 from the ratio test means absolutely nothing. Its like asking a true/false question and getting "football" :) in return, it tells you nothing either way, and it means you are asking the wrong answer-giver (in this case the ratio test). To demonstrate this, Chaz knows $\frac{1}{n^2}$ converges and gets a 1 from the rat. test. However, we know $\frac{1}{n}$ diverges, but it also gets a 1 from the rat. test; $\lim_{n\to\infty}\left|\frac{\frac{1}{n+1}}{\frac{1}{n}}\right| = \lim_{n\to\infty}\frac{n}{n+1} = \lim_{n\to\infty}\frac{n/n}{n/n+1/n} = 1$, so we know that a 1 from the ratio test means nothing for convergence or divergence, because it would be telling you the same thing for a series we know is convergent and a series we know is divergent.

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Very well done!

9.Find the radius of convergence of the power series $\sum_{n=1}^{\infty}(-1)^n \frac{x^n}{n^2}$.

RAT TEST

$(-1)^n \frac{x^n}{n^2}$ ↑ because of abs. value this drops out

$\lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{(n+1)^2}}{\frac{x^n}{n^2}}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)^2}\cdot\frac{n^2}{x^n}\right| = \lim_{n\to\infty}\left|\frac{x\cdot n^2}{n^2+2n+1}\right| = \lim_{n\to\infty}\left|\frac{x\cdot 1}{1+\frac{2}{n}+\frac{1}{n^2}}\right| = \left|\frac{x}{1}\right|$

The ratio test says for the series to converge, $L < 1$. Therefore $|x| < 1$, Interval $(-1 \le x \le 1)$ $L = |x|$

Excellent

Radius of convergence = 1

10. Use the 6^{th} degree polynomial for $\sin(x^2)$ to approximate $\int_0^1 \sin(x^2)\,dx$.

$\sin x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$

10 so $\sin(x^2) = \sum(-1)^n \frac{(x^2)^{2n+1}}{(2n+1)!} = x^2 - \frac{(x^2)^3}{3!} = x^2 - \frac{x^6}{6}$

$\sin(x^2) = x^2 - \frac{x^6}{6}$

$\int_0^1 \sin(x^2)dx = \int_0^1\left(x^2 - \frac{x^6}{6}\right)dx = \int_0^1 x^2\,dx - \int_0^1 \frac{x^6}{6}dx =$

$\left[\frac{x^3}{3}\right]_0^1 - \left[\frac{x^7}{6\cdot 7}\right]_0^1 = \left(\frac{1^3}{3} - \frac{0^3}{3}\right) - \left(\frac{1^7}{42} - \frac{0^7}{42}\right) =$

$\left(\frac{1}{3}\right) - \left(\frac{1}{42}\right) = \frac{14}{42} - \frac{1}{42} = \frac{13}{42}$ Nice

so $\int_0^1 \sin(x^2)\,dx \approx \frac{13}{42} \approx 0.3095$

Extra Credit (This problem can replace your lowest other problem on the test): Find the sum of the series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{8} + \frac{1}{9} + \frac{1}{16} + \frac{1}{27} + \frac{1}{32} + \frac{1}{64} + \frac{1}{81} + \ldots$, where the terms are of the form 1 over all possible natural number exponents on 2 or 3.

$1 + \sum(a_n + b_n) = \sum a_n + \sum b_n + 1$

$\sum a_n = \sum_{n=\phi}^{\infty}(1)\frac{1}{2^n}$

$\sum b_n = \sum_{n=1}^{\infty}(1)\frac{1}{3^n}$

Nice

$\sum a_n = \sum_{n=1}^{\infty}\frac{1}{2^n}$

This is a geometric series where $r = \frac{1}{2}$ and $a = \frac{1}{2}$ so

$S_{na} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$

$\sum b_n = \sum_{n=1}^{\infty}\frac{1}{3^n}$

This also is a geometric series where $r = \frac{1}{3}$ and $a = \frac{1}{3}$ so

$S_{nb} = \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$

So $1 + \sum a_n + \sum b_n$

$= 1 + 1 + \frac{1}{2}$

$= 2\frac{1}{2} = \frac{5}{2}$