Exam 1 Calculus 3 10/18/2002

Each problem is worth 10 points. Show adequate justification for full credit. Please circle all answers and keep your work as legible as possible. Watch that first step.

1. Write an equation for the line between the points (-3,2,5) and (1,-2,4).

$v = (1,-2,4) - (-3,2,5)$

$v = \langle 1-(-3), -2-2, 4-5 \rangle \quad v = \langle 4, -4, -1 \rangle$

$r = r_0 + t v \quad \boxed{r = (-3,2,5) + t\langle 4,-4,-1 \rangle}$

$$\boxed{\begin{aligned} x &= -3 + 4t \\ y &= 2 - 4t \\ z &= 5 - t \end{aligned}}$$

2. Convert the point $(1,-1,\sqrt{2})$ from rectangular to

(a) cylindrical coordinates.

$(1,-1,\sqrt{2})$ $\quad \tan\theta = \frac{y}{x}$ $\quad z = z$

$r^2 = x^2 + y^2$ $\quad \theta = \tan^{-1} \frac{-1}{1}$ $\quad z = \sqrt{2}$

$r = \sqrt{1+1}$ $\quad \theta = -\frac{\pi}{4}$

$r = \sqrt{2}$ $\quad \boxed{\left(\sqrt{2}, -\frac{\pi}{4}, \sqrt{2}\right)}$

(b) spherical coordinates.

$\rho^2 = x^2 + y^2 + z^2$ $\quad z = \rho\cos\phi$ $\quad y = \rho\sin\phi\sin\theta$

$\rho = \sqrt{1+1+2}$ $\quad \sqrt{2} = 2\cos\phi$ $\quad -1 = 2\sin\frac{\pi}{4}\sin\theta$

$\rho = \sqrt{4} = \pm 2$ $\quad \cos\phi = \frac{\sqrt{2}}{2}$ $\quad \theta = -\frac{\pi}{4}$

$\phi = \cos^{-1}\frac{\sqrt{2}}{2}$

$\phi = \frac{\pi}{4}$

$$\boxed{\left(2, -\frac{\pi}{4}, \frac{\pi}{4}\right)}$$

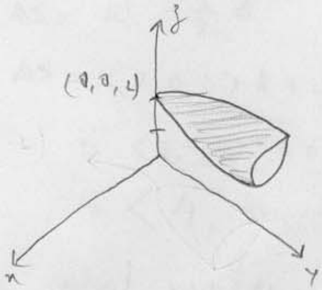
3. Transform the equation $x^2 + z^2 + 4 = y + 4z$ to standard form, identify the quadric surface it represents, give the coordinates of any vertex or vertices, and visualize it shaded in lavender.

$x^2 + z^2 - 4z + 4 = y$

$\Rightarrow x^2 + (z-2)^2 + 4 - 4 = y$

$\Rightarrow x^2 + (z-2)^2 = y$

It is a paraboloid opening in the y-direction with vertex at $(0, 0, +2)$.



Great

4. Show that the length of the conic spiral $\mathbf{r}(t) = \langle t\cos t, t\sin t, t\rangle$ between the points $(0,0,0)$ and $(4\pi, 0, 4\pi)$ is given by $\int_0^{4\pi} \sqrt{2+t^2}\, dt$.

$\vec{r}(t) = \langle t\cos t, t\sin t, t\rangle$ $\quad (0,0,0)$ $\quad (4\pi, 0, 4\pi)$

limits of integration

$t\cos t = 0$
$t\sin t = 0$ when $t=0$
$t = 0$

Yes!

$t\cos t = 4\pi$
$t\sin t = 4\pi$ when $t = 4\pi$
$t = 4\pi$

$\vec{r}'(t) = \langle \cos t - t\sin t, \sin t + t\cos t, 1\rangle$

Arc Length $= \int_0^{4\pi} \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + (1)^2}\, dt$

$= \int_0^{4\pi} \sqrt{\cos^2 t \cancel{- 2t\sin t\cos t} + t^2\sin^2 t + \sin^2 t \cancel{+ 2t\cos t\sin t} + t^2\cos^2 t + 1}\, dt$

$(\cos t - t\sin t)(\cos t - t\sin t) = \cos^2 t - 2t\sin t\cos t + t^2\sin^2 t$

$= \int_0^{4\pi} \sqrt{\cos^2 t + \sin^2 t + t^2(\cos^2 t + \sin^2 t) + 1}\, dt$

$= \int_0^{4\pi} \sqrt{1 + t^2 + 1}\, dt = \int_0^{4\pi} \sqrt{2 + t^2}\, dt$

Very nice!

5. Wile E. Coyote leaps off a ledge as the road runner zips past below him on a perfectly level road. Wile's Acme jetpack propels him in the direction of the positive y axis with an acceleration of of $20 m/s^2$, and gravity in cartoons is $5 m/s^2$ in the direction of the negative z axis, so his acceleration is given by the vector <0, 20, -5>. Wile's jump was from 10 meters above the origin with an initial velocity of 2m/s along the positive x axis, so his initial velocity is given by the vector <2, 0, 0> and his initial position by the vector <0, 0, 10>. What is Wile's position after two seconds?

@ $t=0$ $z=10$ m   acceleration vector $= \langle 0, 20, -5 \rangle = \ddot{r}$

$V_x = 2$ m/s   initial velocity vector $= \langle 2, 0, 0 \rangle$

Initial position vector $\langle 0, 0, 10 \rangle$

antiderivative acceleration = velocity

$$\int \ddot{r} = \langle C_1, 20t + C_2, -5t + C_3 \rangle = \text{velocity}$$

given initial velocity $\longrightarrow \langle 2, 20t, -5t \rangle = $ velocity

Antiderivative velocity = position

$$\int \text{velocity} = \langle 2t + C_4, 10t^2 + C_5, -\tfrac{5}{2}t^2 + C_6 \rangle$$

Excellent

given initial position $\longrightarrow \langle 2t, 10t^2, -\tfrac{5}{2}t^2 + 10 \rangle = $ position

@ $t = 2$ position $= \langle 2(2), 10(2)^2, -\tfrac{5}{2}(2)^2 + 10 \rangle = \langle 4, 40, 0 \rangle$

6. Show that for any vectors **a** and **b**, the vector **b** is perpendicular to **a**×**b**.

Assume vector $a = \langle a_1, a_2, a_3 \rangle$ and vector $b = \langle b_1, b_2, b$

Then $a \times b = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$

For two vectors to be $\perp$, their dot products must be zero, so if b is $\perp$ to $a \times b$, then $b \cdot (a \times b) = ($

$b \cdot (a \times b) = \langle b_1, b_2, b_3 \rangle \cdot \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle = b_1(a_2 b_3 - a_3 b_2) + b_2(a_3 b_1 - a_1 b_3) + b_3(a_1 b_2 - a_2 b_1) = a_2 b_1 b_3 - a_3 b_1 b_2 + a_3 b_1 b_2 - a_1 b_2 b_3 + a_1 b_2 b_3 - a_2 b_1 b_3 = 0$, so b is $\perp$ to $a \times b$

Wonderful!

7. Muffy is a calculus student at E.S.U., and she's having trouble with lines in $\mathbb{R}^3$. Muffy says "I, like, totally can't get any of the answers in the back of the book. I can, like, figure out what the little vector-thingies are, and they say, like, what points to have on the line, but then when I put it together it's not what the solutions manual says. My Daddy hired a tutor to help me and he says I did everything okay, but I don't think that can be right because it's not what's in the book. I mean, like, it's math, so there's always a right answer, right?"

Explain to Muffy, in terms she can understand, **at least three** factors that could lead to correct equations for the same line which still don't look the same.

(1) The direction vectors can be scaler multiples of one another.

One of the neat properties of a vector is that they can be different lengths or even point in different directions and still be equivalent.

Really excellent answer!

For instance if we take the vector $\vec{a} = \langle x_1, y_1, z_1 \rangle$ and cut it in half, $\frac{1}{2}\vec{a} = \langle \frac{1}{2}x_1, \frac{1}{2}y_1, \frac{1}{2}z_1 \rangle$, we still have the same direction vector, it just takes twice as long to get to the same point on the line

(2) Different starting points

Using the vector equation of a line $\vec{r} = \vec{r}_0 + t\vec{r}_d$; Starting at a different point on the line will yield a different appearing equation, but, they are still equivalent.

(3) Different notation.

As there are three different methods of writing a line in 3-D, it is entirely possible the book may be offering the answer in one form, say, parametric or symmetric form, whereas you've given your answer in vector notation.

It helps to be flexible. Yes!

8. Find an equation for the plane containing the line x = 2 + t, y = 3 - t, z = 4 - 2t and passing through the point (5, -2, 3).

$x = 2 + t$
$y = 3 - t$
$z = 4 - 2t$

$\vec{v}_1 = \langle 1, -1, -2 \rangle$ is a vector in the plane

$(2,3,4)$ and $(5,-2,3)$ are points in the plane

so $\vec{v}_2 = \langle 3, -5, -1 \rangle$ is a vector connecting the points, is in the pl

* cross $\vec{v}_1$ and $\vec{v}_2$ to find a vector normal ($\perp$) to the plane:

$$\vec{v}_1 \times \vec{v}_2 = \langle 1,-1,-2 \rangle \times \langle 3,-5,-1 \rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & -2 \\ 3 & -5 & -1 \end{vmatrix} = (1\vec{i} - 6\vec{j} - 5\vec{k}) - (10\vec{i} - 1\vec{j} - 3\vec{k})$$

$$= \langle -9, -5, -2 \rangle$$

so $\vec{n} = \langle -9, -5, -2 \rangle$ is normal vector, so:

Great Job!

$$-9(x-5) - 5(y+2) - 2(z-3) = 0$$

9. Show that a circle with a radius of 3 has a curvature of 1/3.

$x^2 + y^2 = r \Leftarrow$ Circle w/ $z=0$

$x^2 + y^2 = 9$

$x = r\cos\theta$
$y = r\sin\theta$

$x = 3\cos\theta$
$y = 3\sin\theta$

$r = \langle 3\cos\theta, 3\sin\theta, 0 \rangle$

$r' = \langle -3\sin\theta, 3\cos\theta, 0 \rangle$

$r'' = \langle -3\cos\theta, -3\sin\theta, 0 \rangle$

Excellent

$$r' \times r'' = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3\sin\theta & 3\cos\theta & 0 \\ -3\cos\theta & -3\sin\theta & 0 \end{vmatrix} = 0\vec{i} + 0\vec{j} + 9\sin^2\theta - (0\vec{i} + 0\vec{j} + -9\cos^2\theta)$$

$$\langle 0, 0, 9(\sin^2\theta + \cos^2\theta) \rangle = \langle 0, 0, 9 \rangle$$

$$|r' \times r''| = \sqrt{0^2 + 0^2 + 9^2} = 9$$

$r' = \langle -3\sin\theta, 3\cos\theta, 0 \rangle$

$\sqrt{9\sin^2\theta + 9\cos^2\theta + 0} = \sqrt{9(\sin^2\theta + \cos^2\theta)}$

$\sqrt{9} = 3$

$$\frac{9}{3^3} = \frac{9}{27} = \boxed{\frac{1}{3}}$$

10. On the problem set we saw that the set of points equidistant from two fixed points forms a plane. Find the set of points whose distance from (0,0,0) is equal to twice their distance from (3,0,0).

The distance from $(0,0,0)$ to (x,y,z) is given by $\sqrt{(x-0)^2+(y-0)^2+(z-0)^2}$
and the distance from $(3,0,0)$ to (x,y,z) is given by $\sqrt{(x-3)^2+(y-0)^2+(z-0)^2}$
So if we want the first to be twice the second, we have

$$\sqrt{(x-0)^2+(y-0)^2+(z-0)^2} = \sqrt{(x-3)^2+(y-0)^2+(z-0)^2} \cdot 2$$

squaring:
$$x^2+y^2+z^2 = (x^2-6x+9+y^2+z^2)\cdot 2^2$$
$$x^2+y^2+z^2 = 4x^2-24x+36+4y^2+4z^2$$
$$0 = 3x^2-24x+36+3y^2+3z^2$$
$$0 = x^2-8x+12+y^2+z^2$$

completing the square:
$$16-12 = x^2-8x+16+y^2+z^2$$
$$4 = (x-4)^2+y^2+z^2$$

Which we can recognize as a circle with radius 2 centered at $(4,0,0)$.

Extra Credit (up to 5 points possible): Given the lines represented by $\mathbf{r}_1(t)=<1+t, t-3, -2t>$ and $\mathbf{r}_2(t)=<5-3t, 3t, 7>$, find a vector equation for the line which intersects both of these lines and is perpendicular to each of them.