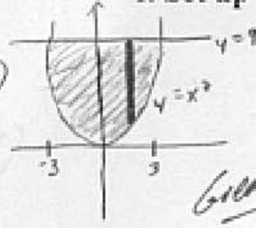
Exam 4   Calculus 3   11/15/2002

Each problem is worth 10 points. Show adequate justification for full credit. Please circle all answers and keep your work as legible as possible. Cultural cliches are offered for critique only.

1. **Set up** a double integral for the area of the ebon region between $y = x^2$ and $y = 9$.

10



$$\int_{-3}^{3}\int_{x^2}^{9} 1 \; dy\, dx$$

$$\iint_R f(x,y)\, dA$$

Great

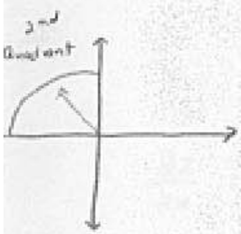
$x^2 = 9$

$x = \pm 3$

2. **Set up** a double integral in polar coordinates for the area of the second-quadrant portion of a puce circle centered at the origin with radius 7.

10

$x^2 + y^2 = 49$ $r = 7$ θ goes from $\pi/2$ to π

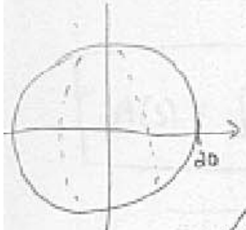


Yes $$\int_{\pi/2}^{\pi}\int_{0}^{7} 1 \cdot r \cdot dr\, d\theta$$

3. **Set up** a triple integral in spherical coordinates for the volume of the first-octant part of a magenta sphere centered at the origin with radius 20.

10

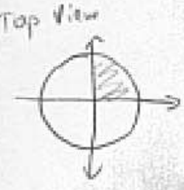
$\rho = 0$ to 20 $\theta = 0$ to $\pi/2$ $\phi = 0$ to $\pi/2$



$$\int_{0}^{\pi/2}\int_{0}^{\pi/2}\int_{0}^{20} 1 \cdot \rho^2 \sin\phi \; d\rho\, d\phi\, d\theta$$

Good

Top View



$x^2 + y^2 + z^2 = 400$

$\rho^2 = x^2 + y^2 + z^2$

$\rho^2 = 400$ $\rho = 20$

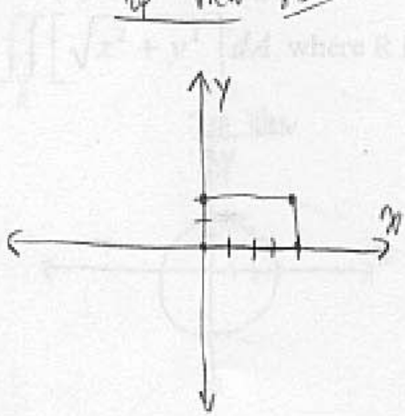
4. Find the Jacobian for the transformation x = u + 4v, y = 3u - 2v.

$$\begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 4 \\ 3 & -2 \end{vmatrix} = -2 - 12 = \boxed{-14}$$

Great

5. Set up an iterated integral for the **surface area** of the part of the cylinder $y^2 + z^2 = 9$ that lies above the rectangle with vertices (0,0), (4,0), (0,2), and (4,2).

Top View yes



$z^2 = 9 - y^2$

$z = \sqrt{9-y^2}$

$f_x = 0$

$f_y = \dfrac{1}{\cancel{2}\sqrt{9-y^2}} \cdot -\cancel{2}y$

$f_y = \dfrac{-y}{\sqrt{9-y^2}}$

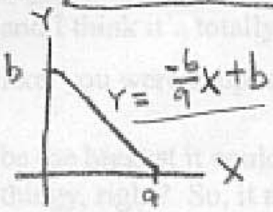
$$\int_0^4 \int_0^2 \sqrt{1 + (f_x)^2 + (f_y)^2}\; dy\, dx$$

Good Job!

$$\int_0^4 \int_0^2 \sqrt{1 + (0)^2 + \left(\frac{-y}{\sqrt{9-y^2}}\right)^2}\; dy\, dx$$

$$\boxed{\int_0^4 \int_0^2 \sqrt{1 + \frac{y^2}{9-y^2}}\; dy\, dx}$$

6. **Set up** iterated integrals for the **center of mass** of a triangular lamina of density $\rho(x,y) = k$ with vertices (0,0), (a,0), and (0,b).



$$m = \int_0^a \int_0^{-\frac{b}{a}x+b} k\, dY\, dx = \text{mass}$$

$$\bar{X} = \frac{1}{m}\int_0^a \int_0^{-\frac{b}{a}x+b} x k\, dY\, dx$$

note input of mass

$$\bar{Y} = \frac{1}{m}\int_0^a \int_0^{-\frac{b}{a}x+b} Y k\, dY\, dx$$

Well done!

center of mass $= (\bar{X}, \bar{Y})$

7. If [x] denotes the greatest integer less than or equal to x, **evaluate** the integral $\iint_R \left[\sqrt{x^2+y^2}\right] dA$ where R is the disk with center at the origin and radius 3.

greatest integer less than or equal to $\sqrt{x^2+y^2}$

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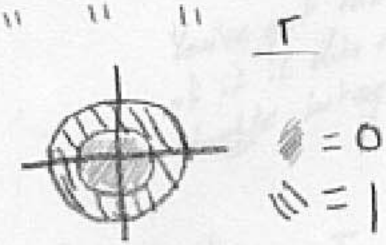
$$\int_0^{2\pi}\int_0^3 [r]\, r\, dr\, d\theta$$



$$\int_0^{2\pi}\int_0^1 0\, r\, dr\, d\theta + \int_0^{2\pi}\int_1^2 r\, dr\, d\theta + \int_0^{2\pi}\int_2^3 2r\, dr\, d\theta$$

Beautifully done!

$$0 + \int_0^{2\pi} \left.\frac{r^2}{2}\right]_1^2 d\theta + \int_0^{2\pi} \left. r^2 \right]_2^3 d\theta$$

$$\int_0^{2\pi} \frac{3}{2}\, d\theta + \int_0^{2\pi} 5\, d\theta$$

$$\frac{6\pi}{2} + 10\pi = 13\pi$$

8. Muffy is a calculus student at E.S.U., and she's having trouble with multiple integrals. Muffy says "Ohmygod, I so totally failed my Calc exam. There were totally impossible problems on it, and I think it's totally bad, and my daddy is going to sue the school. There was this one that was like, you were supposed to find the region-thingy for this integral $\iiint_D \left(1-x^2-y^2-z^2\right)dV$ to be the biggest it could be, and I said, like, obviously it's bigger if you do it for a bigger region-thingy, right? So, it must be biggest if you have D be like all of $\mathbb{R}^3$, right? But I got no points, so Daddy's going to get the professor fired.

Clarify for Muffy, in terms she can understand, how she should think about a problem like this, and what region D in fact maximizes the given integral.

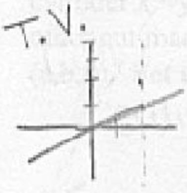
See Muffy, the problem is that this is a triple integral. The integrand (stuff on the inside) is the density. Yes! In this case, if you have a large region, the denisty is negitive far out and you get a negitive result. The only time that the funtion is positive is within the shere of radius 1. Any values for x, y, or z greater than that result in a negitive density

Yes!

$2\cos(x^2)\cdot z^{-1/2}$

9. Find the exact value of $\displaystyle\int_0^4\int_0^1\int_{2y}^2 \frac{2\cos\left(x^2\right)}{\sqrt{z}}\,dx\,dy\,dz$.

T.V.



$$\int_{x=0}^{x=2}\int_{y=0}^{y=\frac{1}{2}x}\int_{z=0}^{z=4} \frac{2\cos(x^2)}{\sqrt{z}}\,dz\,dy\,dx = \int_{x=0}^{x=2}\int_{y=0}^{y=\frac{1}{2}x}\left(4\cos(x^2)z^{1/2}\Big|_0^4\right) dy\,dx =$$

$y=\frac{1}{2}x$

$$\int_{x=0}^{x=2}\int_{y=0}^{y=\frac{1}{2}x} 8\cos(x^2)-0\,dy\,dx = \int_{x=0}^{x=2}\int_{y=0}^{y=\frac{1}{2}x} 8\cos(x^2)\,dy\,dx =$$

$$\int_{x=0}^{x=2} 8\cos(x^2)\,y\Big|_0^{\frac{1}{2}x}\,dx = \int_0^2 4x\cos(x^2)-0\,dx =$$

$$\int_0^2 4x\cos(x^2)\,dx = \int_0^2 \cos u\cdot 2\,du = 2\int_0^2 \cos u\,du =$$

$u=x^2$
$du=2x\,dx$

$$2\left[\sin u\right] = 2\left[\sin(x^2)\Big|_0^2 = 2\sin 4 - 2\overset{0}{\sin 0} =$$

$$\boxed{2\sin 4}$$

Excellent

10. Pat the mathematician has gone completely off the deep end and is now leaving the mainstream catering business for the highly specialized sausage sculpture niche business. Pat is trying to figure out the volume of novel sausage sections. Suppose the sausage is shaped like the cylinder $x^2+y^2=1$, with one cut made perpendicular to the cylinder along the plane $z=0$ and the other cut made along a paraboloid shaped like $z = x^2 + y^2$ but translated to have its vertex at (a,b,0). **Set up** an iterated integral for the resulting volume and **evaluate** it.

$x^2+y^2 = r^2 = 1,\ r = \pm 1,\ z = 0 \to r^2$

under this: use double

$z = (x-a)^2 + (y-b)^2$

within this:

$x = r\cos\theta$

$y = r\sin\theta$

$(r\cos\theta - a)(r\cos\theta - a)$

$r\sin\theta - b$

$\theta \quad r$

$$\int_0^{2\pi}\int_0^1 (x-a)^2 + (y-b)^2\, r\,dr\,d\theta$$

$$\int_0^{2\pi}\int_0^1 (r^2\cos^2\theta + a^2 + r^2\sin^2\theta + b^2 - 2ar\cos\theta - 2br\sin\theta)\, r\,dr\,d\theta$$

suppose 0,0,0;

$$\int_0^{2\pi}\int_0^1 (r^2(\cos^2\theta + \sin^2\theta) + a^2 + b^2 - 2ar\cos\theta - 2br\sin\theta)\, r\,dr\,d\theta$$

$$\iint (r^2 + a^2 + b^2 - 2ar\cos\theta - 2br\sin\theta)\, r\,dr\,d\theta$$

$$\iint (r^3 + a^2 r + b^2 r - 2ar^2\cos\theta - 2br^2\sin\theta)\,dr\,d\theta$$

$$\int \left(\tfrac{1}{4}r^4 + \tfrac{a^2}{2}r^2 + \tfrac{b^2}{2}r^2 - \tfrac{2}{3}ar^3\cos\theta - \tfrac{2}{3}br^3\sin\theta\right]_0^1 d\theta$$

$$\int_0^{2\pi} \left(\tfrac{1}{4} + \tfrac{a^2}{2} + \tfrac{b^2}{2} - \tfrac{2}{3}a\cos\theta - \tfrac{2}{3}b\sin\theta\right) d\theta$$

$$\left[\left(\tfrac{1}{4} + \tfrac{a^2}{2} + \tfrac{b^2}{2}\right)\theta - \tfrac{2}{3}a\sin\theta + \tfrac{2}{3}b\cos\theta\right]_0^{2\pi}$$

$$\left[\left(\tfrac{\pi}{2} + a^2\pi + b^2\pi\right) - 0 + \tfrac{2}{3}b\right] - \left[\tfrac{2}{3}b\right] = \frac{\pi}{2} + a^2\pi + b^2\pi$$