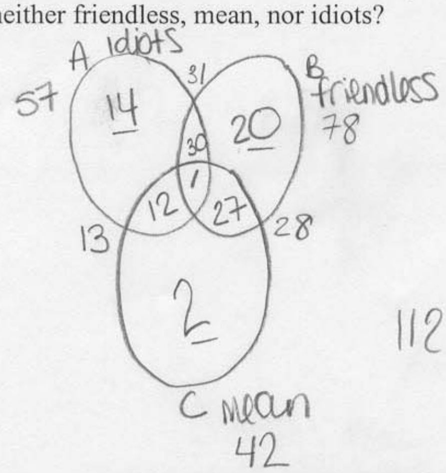
Exam 1 Real Analysis 1 10/4/2002

Each problem is worth 10 points. Show adequate justification for full credit. Don't panic.

A

1. Among the 112 faculty at a certain small liberal arts college, it is discovered that 57 are idiots, 78 are friendless, and 42 are mean people. Further examination reveals that 31 are both idiots and friendless, 13 are both idiots and mean, and 28 are both mean and friendless. If the only faculty member who is a friendless, mean, idiot is named Jon, then how many faculty members are neither friendless, mean, nor idiots?

10



112

A=57

B=78

C=42

112-106 = 6 are not friendless, mean a idiots

2. Give an example of an odd function (you need $\boxed{\text{not}}$ prove that it's odd, so long as it is).

$f(-x) = -f(x)$ is what it means to be an odd function

Whew! :)

$-x^3$ is an odd function

Excellent

$f(-x)^3 = -f(x)^3$

3. State the definition of *convergence* of a sequence $\{a_n\}$.

a sequence $\{a_n\}$ converges to a real number A iff for any $\varepsilon > 0$ there exists n^* such that

$|a_n - A| < \varepsilon$ for any $n > n^*$

Perfect

4. State the definition of an *increasing* sequence.

A sequence $\{a_n\}$ is said to be increasing iff for any $n, m \in \mathbb{N}$ & $n \leq m$, we have $a_n \leq a_m$.

Great

5. Prove that if n is a natural number for which n^2 is odd, then n is also odd.

$n \in \mathbb{N}$ n^2 is odd $(2n+1)^2$ so $n = (2r+1)$

Proof: S'pose n is not odd but even and n^2 is odd

so $n = 2r$

and $n^2 = (2r)^2 = 4r^2$

but $4r^2$ is not odd, it is even because $4r^2 = 2(2r^2)$ where $2r^2$ could $= q$ so $n^2 = 2q$ which is even.

Thus if n^2 is odd, n must also be odd.

Excellent

6. Prove that the sum of the first n odd natural numbers is n^2.

$$\sum_{i=1}^{n} 2(i-1)+1 = n^2 \leftarrow \text{prove this}$$

Let's try to prove this by induction

First lets try $n=1$

$$\sum_{i=1}^{1} 2(i-1)+1 = 2(1-1)+1 = 1^2$$

$$2(0)+1 = 1^2$$

$$1 = 1 \text{ yes!}$$

Now let's assume $\sum_{i=1}^{k} 2(i-1)+1 = k^2$ is true

Prove $\sum_{i=1}^{k+1} 2(i-1)+1 = (k+1)^2$

$$\left(\sum_{i=1}^{k} 2(i-1)+1\right) + 2((k+1)-1)+1 = (k+1)^2$$

$$\left(\sum_{i=1}^{k} 2(i-1)+1\right) + 2k+1 = k^2+2k+1$$

These two terms are equal by my inductive hypothesis so we are left with

$$2k+1 = 2k+1$$

which is true so I have proven

$$\sum_{i=1}^{n} 2(i-1)+1 = n^2 \quad \text{yea!}$$

Really nice job!

$\frac{1}{\sqrt{n}} < \varepsilon$

$\frac{1}{\varepsilon} < \sqrt{n}$ Yes

$\frac{1}{\sqrt{\varepsilon^2}} < n$

7. Prove from the definition that $\left\{\frac{1}{\sqrt{n}}\right\}$ converges to 0.

$|a_n - A| < \varepsilon$ for $n > n^*$

Proof: we are given an $\varepsilon > 0$ and we want $n > n^*$

let $n^* = \frac{1}{\varepsilon^2}$.

so since we want $n > n^*$

$\frac{1}{\varepsilon^2} < n$

$\Leftrightarrow \frac{1}{\varepsilon} < \sqrt{n}$

$\Leftrightarrow \frac{1}{\sqrt{n}} < \varepsilon$

since $\frac{1}{\sqrt{n}}$ will be positive because we only know $\sqrt{\ }$ of a positive #

$\Leftrightarrow \left|\frac{1}{\sqrt{n}}\right| < \varepsilon$

$\left|\frac{1}{\sqrt{n}} - 0\right| < \varepsilon$ This, according to def. of convergence, means that sequence $a_n = \frac{1}{\sqrt{n}}$ and converges to 0. ∎

Wonderful Job!

8. Prove or give a counterexample: If $\{a_n\}$ is a sequence which diverges to $+\infty$ and $\{b_n\}$ is another sequence, then $\{a_n b_n\}$ diverges to $+\infty$.

given $\{a_n\} \to +\infty$ & $\{b_n\}$ is another sequence then $\{a_n b_n\}$ diverges to $+\infty$

let's say $\{a_n\} = \{n^2\}$ which clearly diverges to $+\infty$

if $\{b_n\} = \{-n\}$ then $\{a_n b_n\} = \{-n^3\}$ which diverges to $\underline{-}\infty$ or if $\{b_n\} = \{(-1)^n\}$ the sequence $\{a_n b_n\} = \{(-1)^n n^2\}$ which would oscilate.

so I have given two counter example to the statement above which disproves it.

Very nicely done!

9. Using some or all of the axioms:

(A1) (Closure) $a+b$, $a \cdot b \in \mathbb{R}$ for any $a, b \in \mathbb{R}$. Also, if $a, b, c, d \in \mathbb{R}$ with $a=b$ and $c=d$, then $a+c=b+d$ and $a \cdot c=b \cdot d$.

(A2) (Commutative) $a+b = b+a$ and $a \cdot b = b \cdot a$ for any $a, b \in \mathbb{R}$.

(A3) (Associative) $(a+b)+c = a+(b+c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for any $a, b, c, \in \mathbb{R}$.

(A4) (Additive identity) There exists a zero element in $\mathbb{R}$, denoted by 0, such that $a+0 = a$ for any $a \in \mathbb{R}$.

(A5) (Additive inverse) For each $a \in \mathbb{R}$, there exists an element $-a$ in $\mathbb{R}$, such that $a + (-a) = 0$.

(A6) (Multiplicative identity) There exists an element in $\mathbb{R}$, which we denote by 1, such that $a \cdot 1 = a$ for any $a \in \mathbb{R}$.

(A7) (Multiplicative inverse) For each $a \in \mathbb{R}$ with $a \neq 0$, there exists an element in $\mathbb{R}$ denoted by $\frac{1}{a}$ or a^{-1}, such that $a \cdot a^{-1} = 1$.

(A8) (Distributive) $a \cdot (b+c) = (a \cdot b)+(a \cdot c)$ for any $a, b, c \in \mathbb{R}$.

(A9) (Trichotomy) For $a, b \in \mathbb{R}$, exactly one of the following is true: $a=b$, $a < b$, or $a > b$.

(A10) (Transitive) For $a, b \in \mathbb{R}$, if $a < b$ and $b < c$, then $a < c$.

(A11) For $a, b, c \in \mathbb{R}$, if $a < b$, then $a + c < b + c$.

(A12) For $a, b, c \in \mathbb{R}$, if $a < b$ and $c > 0$, then $ac < bc$.

Prove that if $a, b \in \mathbb{R}$, then $a < b$ if and only if $-a > -b$. Be explicit about which axioms you use.

Proof pt. 1: If $-a > -b$, then $a < b$

$-a+a > -b+a$ A4 well, 11...

$0 > -b+a$ A2

$b+0 > (-b+a+b$

$b+0 > -b+b+a$ A4 Yes!

$\underline{b > a}$

Proof pt. 2: If $a < b$, then $-a > -b$.

$a < b$

$-a+a < b+-a$ A4

$0 < b+-a$

$-b+0 < b+-a+-b$ A4 A2

$-b < b+(-b)+(a)$ A4

$\underline{-b < -a}$

Proof both way because iff $\therefore$ correct ~~#~~ Very nice job!

10. Prove that if $\{a_n\}$ converges to 0, then $\{(a_n)^2\}$ converges to 0.

Proof: If $\{a_n\}$ converges to 0 then there is an n^* such that

$|a_n - 0| < \varepsilon$ for all $n \geq n^*$

so $a_n < \varepsilon$

Suppose $\{(a_n)^2\}$ does converge to 0. Then there exists an n^* such that

$|(a_n)^2 - 0| < \varepsilon$ for all $n \geq n^*$

This would mean

$(a_n)^2 < \varepsilon$

$a_n < \sqrt{\varepsilon}$

but we know $\sqrt{\varepsilon} < \varepsilon$ Yes - nice clever way to handle this.

so $a_n < \sqrt{\varepsilon} < \varepsilon$ and $\{(a_n)^2\}$ converges to 0 ∎

Extra Credit (this problem can replace your lowest-scoring other problem): Prove that $\sqrt{2}$ is irrational.

This is a proof by contradiction. Assume $\sqrt{2}$ is rational, that is $\sqrt{2} = \frac{p}{q}$ where p & q are relatively prime.

Then $2 = \frac{p^2}{q^2}$

$2q^2 = p^2$ which says p^2 is even which makes p even

So $p = 2k$

$p^2 = 4k^2$

So $2q^2 = 4k^2$

$q^2 = 2k^2$ which says q^2 is even which again makes q even.

$\therefore$ p & q are both even having a common factor of 2. This negates our assumption that p & q were relatively prime.

Hence, $\sqrt{2}$ is irrational. ∎

Very nice! Clear, complete, and clean. Well done!