Exam 2 Calculus 1 11/7/2003

Each problem is worth 10 points. Show adequate justification for full credit. Please circle all answers and keep your work as legible as possible. I promise it's revenue-neutral.

1. If $f(x) = x^3 - 6x + 1$, find the exact locations of the local maximum and minimum points of $f(x)$.

$f'(x): 3x^2 - 6 = 0 \quad \frac{3x^2}{3} = \frac{6}{3} \quad \sqrt{x^2} = \sqrt{2} \quad x = \pm\sqrt{2}$

$f''(x) = 6x$

$f(x) = (\sqrt{2})^3 - 6(\sqrt{2}) + 1 = 2^{\frac{3}{2}} - 6\sqrt{2} +$

$6\sqrt{2} = +, \sqrt{2} = \text{min}$

$6 \cdot -\sqrt{2} = -, -\sqrt{2} = \text{max}$

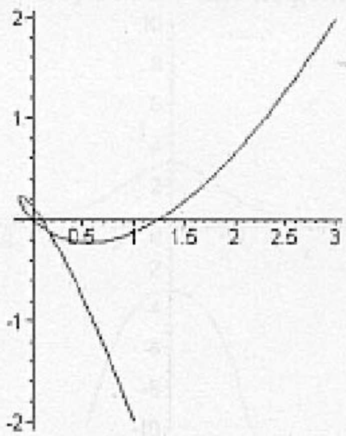
Yes.

$$\boxed{\begin{array}{l} \text{Min } \left(\sqrt{2}, \left((\sqrt{2})^3 - 6\sqrt{2} + 1\right)\right) \\ \text{Max } \left(-\sqrt{2}, \left((-\sqrt{2})^3 + 6\sqrt{2} + 1\right)\right) \end{array}}$$

$\approx (1.41, -4.657)$

$\approx (-1.41, 6.657)$

2. Consider the curve shown below, with parametric equations $x(t) = 2t^2 - t$ and $y(t) = 3t^3 - t$. What are the exact coordinates of the point where a line tangent to the curve is vertical?



$x(t) = 2t^2 - t$

$y(t) = 3t^3 - t$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{9t^2 - 1}{4t - 1}$

$\frac{dy}{dt} = 9t^2 - 1$

Verticle tangent denominator = 0

$\frac{dx}{dt} = 4t - 1$

$4t - 1 = 0$

$4t = 1$

$t = \frac{1}{4}$

Good!

$2(\frac{1}{4})^2 - \frac{1}{4} = x$

$3(\frac{1}{4})^3 - \frac{1}{4} = y$

$\boxed{\text{pnt}: \left(-\frac{1}{8}, -\frac{13}{64}\right)}$

3. **Prove the sum rule for derivatives,** i.e. that $(f+g)'(x) = f'(x) + g'(x)$.

Proof: $(f+g)'(x) = \lim\limits_{h \to 0} \dfrac{(f+g)(x+h) - (f+g)(x)}{h}$

$= \lim\limits_{h \to 0} \dfrac{[f(x+h) + g(x+h)] - [f(x) + g(x)]}{h}$

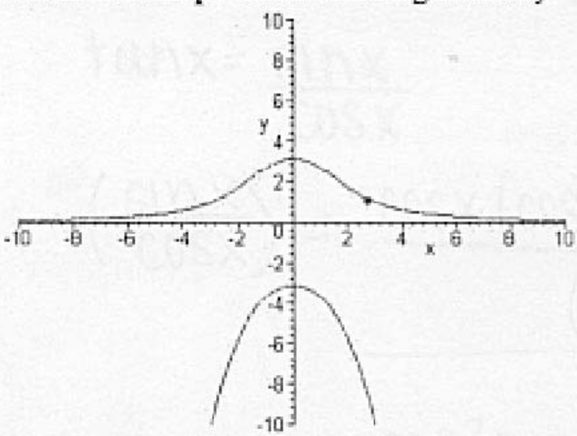
$= \lim\limits_{h \to 0} \dfrac{[f(x+h) - f(x)] + [g(x+h) - g(x)]}{h}$

$= \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h} + \dfrac{g(x+h) - g(x)}{h}$

$= f'(x) + g'(x)$

Nice!

4. **Find the slope** of the line tangent to $x^2y + y^2 = 10$ at the point (3,1).



$x^2y + y^2 = 10$

$2xy + x^2y' + 2yy' = 0$

$-2xy = x^2y' + 2yy'$

$\dfrac{y'(x^2+2y)}{(x^2+2y)} = \dfrac{-2xy}{(x^2+2y)}$

$y' = \dfrac{-2xy}{x^2+2y}$

Great

$y' = \dfrac{-2(3)(1)}{3^2+2(1)}$

$y' = \dfrac{-6}{11}$

5. If an anesthetic used to sedate large mammals during surgeries has a peak effectiveness of 100, which occurs 30 minutes after the drug is administered, find a curve of the form $f(t) = ate^{-bt}$ to represent the effectiveness of the sedative t minutes after administration.

We have, $f(t) = ate^{-bt}$

$$f'(t) = a[1\cdot e^{-bt} + t\cdot e^{-bt}\cdot -b]$$

$$= a[e^{-bt} - bte^{-bt}]$$

or, $0 = ae^{-bt}(1-bt)$ $\quad (\because a \neq 0 \ \& \ e^{-bt} \neq 0)$

$\therefore 1 - bt = 0$

or, $t = \frac{1}{b}$ $\quad \therefore b = \frac{1}{30}$

Now, $f(t) = ate^{-bt}$ $\quad$ or, $100 = a\cdot 30\cdot e^{-b\cdot 30}$ $\quad$ or, $\frac{100}{30} = a\cdot e^{-1}$

Well done

or, $a = \frac{10e}{3}$

Hence, the eqn of the curve is $\boxed{f(t) = \frac{10e}{3}\, t\, e^{-\frac{1}{30}t}}$

6. **Show** that the derivative of tan x is $\sec^2 x$ [remember $\tan x = \sin x/\cos x$ and $\sec x = 1/\cos x$].

$f(x) = \tan = \frac{\sin x}{\cos x}$ $\quad \frac{\underbrace{\sin x}_{f(x)}}{\underbrace{\cos x}_{g(x)}}$

Quotient Rule

$$\frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$

$$= \frac{\cos x \cdot \cos x - \sin x \cdot -\sin x}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

Excellent!

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$\frac{1}{\cos x} = \sec x$ so $\frac{1}{\cos^2 x} = \boxed{\sec^2 x}$

$(\tan x)' = \sec^2 x$ □

7. **Prove** the quotient rule for derivatives [you may use the product rule in doing so if you like].

First, we have to prove: $\left[\frac{1}{g(x)}\right]' = \frac{-g'(x)}{[g(x)]^2}$

$$\left[\frac{1}{g(x)}\right]' = \lim_{h\to 0} \frac{\frac{1}{g(x+h)} - \frac{1}{g(x)}}{h} = \lim_{h\to 0} \frac{\frac{g(x)-g(x+h)}{g(x+h)\cdot g(x)}}{h} =$$

$$= \lim_{h\to 0} \frac{g(x)-g(x+h)}{h\cdot g(x+h)\cdot g(x)} = \lim_{h\to 0} \frac{-[g(x+h)-g(x)]}{h\cdot g(x+h)\cdot g(x)} =$$

$$= \lim_{h\to 0} \frac{-1}{g(x+h)\cdot g(x)} \cdot \frac{g(x+h)-g(x)}{h} = \frac{-1}{[g(x)]^2} \cdot g'(x) = \frac{-g'(x)}{[g(x)]^2} \quad \square$$

Then, using the product rule:

$$\left(\frac{f}{g}\right)'(x) = \left(f\cdot\frac{1}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

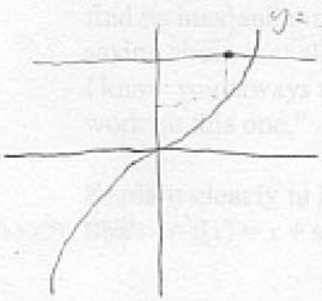
Proof: $\left(f\cdot\frac{1}{g}\right)'(x) = f'(x)\cdot\frac{1}{g(x)} + f(x)\cdot\frac{1}{g'(x)} =$

$$= \frac{f'(x)}{g(x)} + f(x)\left[\frac{-g'(x)}{[g(x)]^2}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

10

Beautifully done.

8. Jim wants to build a $\boxed{\text{rectangular}}$ parking lot on a piece of property shaped like the region between the curve $y = x^3$, the line $y = 8$, and the positive y axis (where the units are measured in increments of how far Jim can walk between puffs on his cigar). What dimensions should the parking lot have in order to maximize its area?



let the right bottom corner be (a, a^3)

$A = a \times (8 - a^3)$

$A = 8a - a^4$

$A' = 8 - 4a^3$

$0 = 8 - 4a^3$

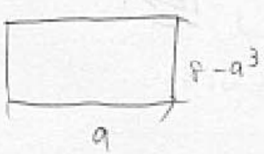
$a = 2^{\frac{1}{3}}$ — ①

Great Job

$A'' = -12a^2$ — ②

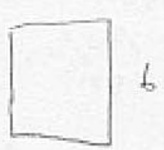
substitute ① into ②

$-12(2^{\frac{1}{3}})^2 \rightarrow$ negative $\rightarrow$ maximum



$a = 2^{\frac{1}{3}}$

the other side is $8 - (2^{\frac{1}{3}})^3 = 6$



$2^{\frac{1}{3}}$ by 6

9. Biff is a business calculus student at a large state university, and he's having some trouble. Biff says "So I hate our professor, and like half the people in the class dropped already but I can't 'cause my dad will kill me. So we've got this test coming up, and the professor was saying something about how this function x plus sin x was really important, and like we're supposed to find its max and min values or something, I don't know. I really couldn't figure out what he was saying about it at all. But I'm pretty sure it's gonna be on the test, so I better know how to do it. I know you always take the derivative and put it equal zero, but I don't know how to make that work on this one."

Explain clearly to Biff what the situation is regarding maximum and minimum values of the function $f(x) = x + \sin x$.

the max + min values can occur when the derivative equals 0. To find out whether it is a max or a min look @ the double derivative. if $f''(x) = +$, then you have a min; if $f''(x) = -$ then max.

$f(x) = x + \sin x$

$f'(x) = 1 + \cos x$

$0 = 1 + \cos x$

$-1 = \cos x$

$x = \pi$ (or $3\pi, 5\pi, 7\pi$ etc)

$f''(x) = -\sin x$

for all crit. pts the $f''(x)$ is 0

Since all of the critical pts make the second derivative zero, we dont really have maxs + mins.

Exactly.

10. If $f(x)$ is a third degree polynomial with a point of inflection at the origin, what can you say about the formula and graph of $f(x)$? Be clear about how you reach your conclusions.

$f(x) = ax^3 + bx^2 + cx + d$

$f'(x) = 3ax^2 + 2bx + c$

$f''(x) = 6ax + 2b \Rightarrow$ when $6ax + 2b = 0$ is the point of inflection @ origin $(0,0)$

$6a(0) + 2b = 0$

$2b = 0$

$\boxed{b = 0}$ Yes!

$\boxed{d = 0}$ because $f(x)$ has to go through the origin, we can't shift it Yes!

$f(x) = ax^3 + cx$

Excellent.