Exam 1 Calculus 3 9/12/2003

Each problem is worth 10 points. Show adequate justification for full credit. Please circle all answers and keep your work as legible as possible. Everything in the book is true, but not everything true is in the book.

1. Write a vector equation for the line passing through the points (-5,7,3) and (1,2,4).

$\vec{A} = \langle -5, 7, 3\rangle \quad \vec{B} = \langle 1, 2, 4\rangle$

a vector in the direction of the line is $\vec{B} - \vec{A}$

$\vec{B} - \vec{A} = \langle 1-(-5), 2-7, 4-3\rangle = \langle 6, -5, 1\rangle$

$\vec{r} = \vec{r}_0 + \vec{v}t \qquad \boxed{\vec{r} = \langle -5, 7, 3\rangle + t\langle 6, -5, 1\rangle}$ Good

2. A koala bear is sitting in a tree at the juncture of two branches with directions given by the vectors 3**i** + 2**j** + 2**k** and **i** - 5**j** + **k**. Find the angle between the branches.

$a \cdot b = |a||b| \cos\theta \qquad \underset{a}{\langle 3, 2, 2\rangle} \quad \underset{b}{\langle 1, -5, 1\rangle}$

$\cos\theta = \dfrac{a \cdot b}{|a||b|} \qquad \langle 3,2,2\rangle \cdot \langle 1,-5,1\rangle = 3 + -10 + 2 = -5$

$|a| = \sqrt{9+4+4} = \sqrt{17} \quad |b| = \sqrt{1+25+1} = \sqrt{27}$

$\cos\theta = \dfrac{-5}{\sqrt{17 \cdot 27}}$

$\cos\theta = \dfrac{-5}{3\sqrt{51}}$

$\theta = 103.5°$

Yes

3. Find the length of the curve $r(t) = < 2 \sin t, 5t, 2 \cos t >$ for $-10 \le t \le 10$.

Arc Length $= \int_a^b \sqrt{(x'^t)^2 + (y'^t)^2 + (z'^t)^2}\, dt$

$r(t) = \langle 2\sin t, 5t, 2\cos t \rangle$

$r'(t) = \langle 2\cos t, 5, -2\sin t \rangle$

$L = \int_{-10}^{10} \sqrt{(2\cos t)^2 + (5)^2 + (-2\sin t)^2}\, dt$

$= \int_{-10}^{10} \sqrt{4\cos^2 t + 4\sin^2 t + 25}\, dt$

$= \int_{-10}^{10} \sqrt{4(\cos^2 t + \sin t) + 25}\, dt$

$= \int_{-10}^{10} \sqrt{4(1) + 25}\, dt$

$= \int_{-10}^{10} \sqrt{29}\, dt$ now integrate

$\sqrt{29}\, t \Big]_{-10}^{10}$

$\sqrt{29}(10) + \sqrt{29}(+10)$

$\boxed{AL = 20\sqrt{29}}$

Nice Job!

4. Find the curvature of $r(t) = <t, t^2, t^3>$ (which is a curve called a *twisted cubic*, which is a pretty funny name) at the point (0,0,0).

$\frac{|r'(t) \times r''(t)|}{|r'(t)|^3}$

$r'(t) = \langle 1, 2t, 3t^2 \rangle$

$r''(t) = \langle 0, 2, 6t \rangle$

$|r'(t)|^3 = (\sqrt{1+4t^2+9t^4})^3$

$r'(t) \times r''(t)$:

$\begin{vmatrix} i & j & k \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix}$

$i\begin{vmatrix} 2t & 3t^2 \\ 2 & 6t \end{vmatrix} - j\begin{vmatrix} 1 & 3t^2 \\ 0 & 6t \end{vmatrix} + k\begin{vmatrix} 1 & 2t \\ 0 & 2 \end{vmatrix}$

$i(12t^2 - 6t^2) - j(6t) + k(2)$

$i6t^2 - j6t + k2$

$|\langle 6t^2, -6t, 2 \rangle|$

$\sqrt{36t^4 + 36t^2 + 4}$

$= \sqrt{4(9t^4 + 9t^2 + 1)}$

$= 2\sqrt{9t^4 + 9t^2 + 1}$

$\frac{|r'(t) \times r''(t)|}{|r'(t)|^3} = \frac{2\sqrt{9t^4+9t^2+1}}{(1+4t^2+9t^4)^{\frac{3}{2}}}$

when $t = 0$ $= \frac{2\sqrt{1}}{1}$

$= 2$

Great

5. Convert the point with cylindrical coordinates $(3, 3\pi/4, 2)$ to

a) Rectangular coordinates

$x = r\cos\theta = 3 \cdot \cos\left(\frac{3\pi}{4}\right) = -\frac{3\sqrt{2}}{2}$

$y = r\sin\theta = 3 \cdot \sin\left(\frac{3\pi}{4}\right) = \frac{3\sqrt{2}}{2}$

$z = z = 2$

a. $\boxed{\left(-\frac{3\pi}{2}, \frac{3\pi}{2}, 2\right)}$

b) Spherical coordinates

$\rho^2 = x^2 + y^2 + z^2,\ \rho = \sqrt{\left(\frac{3\sqrt{2}}{2}\right)^2 + \left(\frac{3\sqrt{2}}{2}\right)^2 + 2^2} = \sqrt{13}$

$\theta = 3\pi/4$ from above

$\phi = \arccos\frac{z}{\rho} = \arccos\frac{2}{\sqrt{13}} = .983$

$\left(\sqrt{13}, \frac{3\pi}{4}, .983\right)$ Yes

6. Write an equation for the plane passing through the point $(0,0,c)$ and containing the vectors $<1,0,m>$ and $<0,1,n>$.

Cross Product :- $\langle 1, 0, m\rangle \times \langle 0, 1, n\rangle = \begin{vmatrix} i & j & k \\ 1 & 0 & m \\ 0 & 1 & n \end{vmatrix}$

$= i(-m) - j(n) + k(1) = \langle -m, -n, 1\rangle$

Normal vector of plane is $\langle -m, -n, 1\rangle$

Eq. of plane :- $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

$\Rightarrow -m(x - 0) - n(y - 0) + 1(z - c) = 0$

Excellent

$\Rightarrow \boxed{-mx - ny + z - c = 0}$

7. Show that for any three-dimensional vectors a and b, a × b must be perpendicular to a.

Let $\vec{a} = \langle a_1, a_2, a_3 \rangle$

$\vec{b} = \langle b_1, b_2, b_3 \rangle$

therefore

$$\langle \vec{a} \times \vec{b} \rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$\hat{i}(a_2b_3 - a_3b_2), -\hat{j}(a_1b_3 - a_3b_1), +\hat{k}(a_1b_2 - a_2b_1)$$

$$\langle a_2b_3 - a_3b_2, +a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$$

Now to show that this is $\perp$ to a

$$\langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle \cdot \langle a_1, a_2, a_3 \rangle$$

$$= (\cancel{a_1a_2b_3} - \cancel{a_1a_3b_2} + \cancel{a_2a_3b_1} - \cancel{a_1a_2b_3} + \cancel{a_1a_3b_2} - \cancel{a_2a_3b_1})$$

$$= 0$$

Exactly.

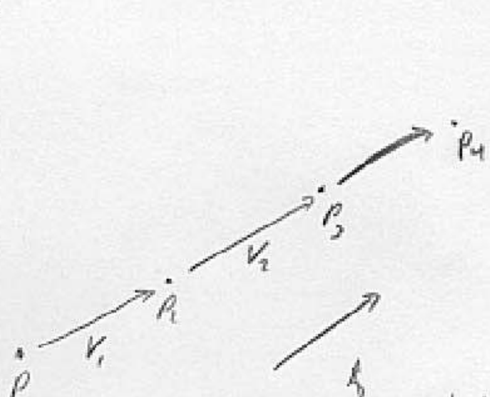
For a dot product to be 0 means that the two quantities must be $\perp$. Hence $\vec{a} \times \vec{b}$ is perpendicular to $\vec{a}$.

8. Biff is a Calc 3 student at a large state university, and he's having some trouble. Biff says "Man, this calc stuff is kicking my butt. There was this problem on our test, and there were four points, and you were supposed to say if they all were on the same line or not, like in three dimensions. I could totally do it if there were three points, 'cause there was an example like that in the book, but they never told us how to do it if there was four points. It's totally unfair, when they haven't even told us how to do it yet. I wrote that it was impossible to determine it if there were four points, because I figured if there were a way, then it would have been in the book, right?"

Explain clearly to Biff a reasonable way to determine whether four points all lie on a common line, or defend his assertion that it's impossible to determine whether such a thing happens.

If three points are on a line this implies that the vector from the first point to the second point is in the same direction as the vector from the second point to the third point and so on. This also means that each of the connecting vectors are parallel. You can determine if the vectors are parallel by using a cross product that yields a zero for the answer or by determining if the unit vectors match. With this process you can continue to see if indeed the 4th point lies on the same line.

Nice way to do it.



note that although it is parallel it isn't necessarily on the line. You must use the points to determine the vector to ensure the vector goes from one point to the next

9. Prove or give a counterexample: The dot product is *associative*, i.e. for any three three-dimensional vectors **a**, **b**and **c**,

$$\mathbf{a} \cdot (\mathbf{b} \cdot \mathbf{c}) = (\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c}$$

~~I Cannot take the dot product with a scalar and a vector.~~

~~b·c = scalar~~

~~a = vector~~

~~a·b = scalar~~

~~c = vector~~

~~In order to do a dot product, two vectors are required.~~

Correct!

It's not associative

Let a = (0,0,1)

Let b = (1,1,1)

Let c = (1,1,1)

$a * (b \cdot c) = (0,0,1) * 3 = (0,0,3)$

$(a \cdot b) * c = 1 * (1,1,1) = (1,1,1)$

Since $a * (b \cdot c) \neq (a \cdot b) * c$,

the dot product is not associative.

Right again!

10. Suppose you want to write an equation for a hyperboloid of one sheet. You want it to open along the z-axis, and have its smallest cross section, a circle with radius 3, where it crosses the plane $z = 1$. You also want it to have a circle with radius 5 for its cross section where it crosses the plane $z = 0$. Find an equation for such a surface.

hyperboloid, one sheet z-axis, general formula $1=\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}$

when $z=1$; circle radius 3 (smallest) $\rightarrow 3^2=x^2+y^2$

$z=1 \to 0$ $\quad 1=\frac{x^2}{9}+\frac{y^2}{9}-\frac{(z-1)^2}{c^2}$

when $z=0$ $\quad 5^2 = x^2+y^2$

$1=\frac{x^2}{9}+\frac{y^2}{9}-\frac{1}{c^2}$

$\frac{9}{c^2}+9 = x^2+y^2$

$25=\frac{9}{c^2}+9$

$16c^2=9$

$c^2=\frac{9}{16}$

$$1=\frac{x^2}{9}+\frac{y^2}{9}-\frac{(z-1)^2}{(9/16)}$$

Beautifully done.