M.B.S.O. 2 Calculus 3 10/1/2003

Each problem is worth 10 points. Show adequate justification for full credit. Please circle all answers and keep your work as legible as possible. It's a slippery slope.

1. State the formal definition of the partial derivative of the function $f(x,y)$ with respect to y.

$$f_y(x,y) = \lim_{h \to 0} \frac{f(x,y+h) - f(y)}{h}$$

Yep.

2. Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 - xy}{x^2 + y^2}$ does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{x^2+y^2}$$

along $x=0$

$$\lim_{(0,y)\to(0,0)} \frac{0^2 - 0y}{0^2+y^2} = \lim_{(0,y)\to(0,0)} \frac{0}{y^2} = 0$$

along $y=0$

$$\lim_{(x,0)\to(0,0)} \frac{x^2 - x(0)}{x^2+0^2} = \lim_{(x,0)\to(0,0)} \frac{x^2}{x^2} = 1$$

$0 \neq 1$

the limits when approached from $x=0$ and $y=0$ do not agree, therefore the limit does not exist

Great

3. If $\mathbf{v} = \langle -5, 12 \rangle$ and $f(x,y) = 4x - xy^2$, what is the directional derivative of f in the direction of $\mathbf{v}$?

$\langle f_x(x,y), f_y(x,y) \rangle \cdot \langle a, b \rangle$

Find the unit vector for $\vec{v}$

$|\langle -5, 12 \rangle| = 13$

$\frac{1}{13}\langle -5, 12 \rangle = \langle -\frac{5}{13}, \frac{12}{13} \rangle$

$f_x = 4 - y^2$

$f_y = -2xy$

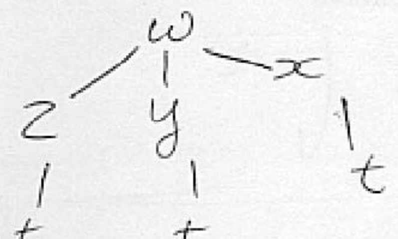
$\langle 4 - y^2, -2xy \rangle \cdot \langle -\frac{5}{13}, \frac{12}{13} \rangle$

or

$(4 - y^2)(-\frac{5}{13}) + (-2xy)(\frac{12}{13})$

Great

4. If $w = f(x,y,z)$, $x = x(t)$, $y = y(t)$, and $z = z(t)$, state the appropriate version of the chain rule for $\frac{dw}{dt}$. **Make it clear which of your derivatives are partials.**



$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial w}{\partial z} \cdot \frac{dz}{dt}$$

partial

W

Good

5. If $f(x,y) = x^3y - y^2$, in which direction is the directional derivative at the point (-1,2) greatest, and what is the value of that directional derivative?

$f_x(x,y) = 3x^2y \qquad f_y(x,y) = x^3 - 2y$

$\nabla f = \langle 3x^2y,\ x^3 - 2y \rangle$

$\nabla f(-1,2) = \langle 3\cdot1\cdot2,\ -1 - (2\cdot2) \rangle$

$\nabla f(-1,2) = \langle \underline{6, -5} \rangle$ direction of greatest increase

$|\nabla f| = \sqrt{6^2 + (-5)^2}$

$= \sqrt{36 + 25}$

$= \underline{\sqrt{61}}$ magnitude

Good

$z = 1 - x - y$

6. Find the point on the plane x + y + z = 1 closest to the point (3, 0, 0).

$d^2 = (x-3)^2 + (y-0)^2 + (z-0)^2$

$d^2 = (x-3)^2 + y^2 + (1-x-y)^2 = f(x,y)$

$f_x(x,y) = 2(x-3) + 2(-1+x+y) = 2x - 6 - 2 + 2x + 2y = \underline{4x + 2y - 8}$

$f_y(x,y) = 2y - 2(1-x-y) = 2y - 2 + 2x + 2y = \underline{2x + 4y - 2}$

$4x + 2y - 8 = 0$

$2x + 4y - 2 = 0$

$4x = 8 - 2y$

$x = 2 - \frac{1}{2}y$

$2(2 - \frac{1}{2}y) + 4y - 2 = 0$

$4 - y + 4y - 2 = 0$

$2 + 3y = 0$

$3y = -2$

$\boxed{y = -\frac{2}{3}}$

$\therefore x = 2 - \frac{1}{2}\left(\frac{-2}{3}\right) = \boxed{\frac{7}{3} = x}$

$z = 1 - \frac{7}{3} - \frac{-2}{3} = -\frac{2}{3}$

$\boxed{\left(\frac{7}{3}, -\frac{2}{3}, -\frac{2}{3}\right)}$

$f_{xx} = 4$

$f_{yy} = 4$

$f_{xy} = 2$

$\therefore D = 16 - 4 = 12$, min

Excellent

7. Bunny is a calc 3 student at a large state university and she's having some trouble. Bunny says "Ohmygod, I am so totally confused by this class. I mean, I can work out a lot of the problems, but I totally don't understand what any of it means. I guess it doesn't really matter, since our exams are all multiple choice, but it really seems like some day I might need to know why some of this stuff works. Like, I totally know that when the question says to find the direction of greatest increase, you figure out the gradient thing and that's the answer. But why? I have no clue, even if I'm getting an A."

Explain clearly to Bunny why the gradient is connected to a direction of greatest increase.

An easy way to see that the gradient is connected to the direction of greatest increase is to see how it fits into the directional derivative. D_u tells us the direction of the rate of change. One form of the equation is

$$D_u = |\text{grad} f||u| \cos\theta.$$

Since u is a unit vector $|u|=1$ and the equation becomes

$$D_u = |\text{grad } f| \cos\theta$$

This equation is maximized when θ, the angle between the gradient and the unit vector is 0 (or there is no difference in the direction of the gradient or the unit vector). When $\theta = 0$ the equation becomes

$$D_u = |\text{grad } f|$$

showing that the gradient is pointing in the direction of greatest increase.

Excellent

8. Find the maximum value(s) of the function $f(x,y) = x^2 + 2y^2$ subject to the constraint $x^2 + y^2 = 4$.

LeGrange Multipliers: $\nabla f(x,y) = \lambda \nabla g(x,y)$

$g = 4$

$\nabla f(x,y) = x^2 + 2y^2$

$f_x(x,y) = 2x$

$f_y(x,y) = 4y$

$\langle 2x, 4y \rangle = \lambda \langle 2x, 2y \rangle$

$g = x^2 + y^2$

$\nabla g(x,y) = y^2 + y^2$

$f_x(x,y) = 2x$

$f_y(x,y) = 2y$

$2x = \lambda 2x$

$0 = \lambda \cdot 2x - 2x$

$0 = -2x(\lambda + 1)$

$x = 0 \quad \lambda = -1$

$4y = \lambda \cdot 2y$

$x^2 + y^2 = 4$

10

Well done!

So, when $x = 0$

$0^2 + y^2 = 4$

$y^2 = 4$

$y = \pm 2$

$(0,2)\ (0,-2)$

when $\lambda = -1$

$4y = -1 \cdot 2y$

$4y = -2y$

$4y + 2y = 0$

$6y = 0$

$y = 0$

$x^2 + 0^2 = 4$

$x = \pm 2$

$(2,0)\ (-2,0)$

$f(0,2) = 0 + 8 = 8$

$f(0,-2) = 0 + 8 = 8$

maximum values

$f(2,0) = 4 + 0 = 4$

$f(-2,0) = 4 + 0 = 4$

9. For which values of b is $f(x,y) = x^2 + bxy + y^2$ a hyperbolic paraboloid?

$$f(x,y) = x^2 + bxy + y^2$$

$$= x^2 + 2 \cdot \frac{b}{2} xy + \left(\frac{b}{2}y\right)^2 - \left(\frac{by}{2}\right)^2 + y^2$$

$$= \left[\left(x + \left(\frac{b}{2}y\right)\right)^2 + y^2\left(1 - \frac{b^2}{4}\right)\right]$$

Always positive

positive if $-2 \le b \le 2$

negative if $b \notin (-2,2)$.

since it's a hyperbolic paraboloid,

$D < 0$ where $D = \frac{4-b^2}{4}$

$\Rightarrow 4 - b^2 < 0$

$\Rightarrow b^2 > 4 \Rightarrow b \in (-\infty, 2) \cup (2, \infty)$

Beautiful Work.