Exam 3 Calculus 3 10/29/2003

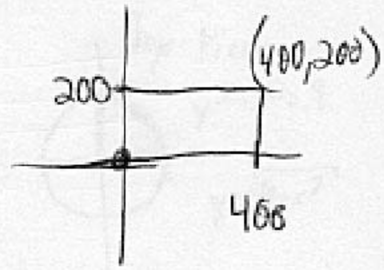
Each problem is worth 10 points. Show adequate justification for full credit. Please circle all answers and keep your work as legible as possible.

$$\boxed{x = \rho\sin\phi\cos\theta \quad y = \rho\sin\phi\sin\theta \quad z = \rho\cos\phi}$$

1. If the function $h(x, y) = -2x + 3y + 2000$ gives the density of field mice per square mile in Kansas (where Kansas is taken to be a rectangle with its lower left corner at the origin and other vertices at (400,0), (400,200), and (0,200)), **write an integral** for the total number of field mice in Kansas.

$$\int_{x=0}^{x=400}\int_{y=0}^{y=200} (-2x+3y+2000)\,dy\,dx$$

Good



2. **Set up** an iterated integral for the area of the region inside $x^2 + y^2 = 9$, outside $x^2 + y^2 =$ above $y = 0$, and below $y = x$.

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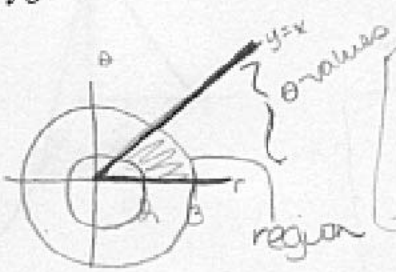
$y = r\cos\theta$



$$\boxed{\int_{\theta=0}^{\theta=\frac{\pi}{4}}\int_{r=2}^{r=3} 1\cdot r\,dr\,d\theta}$$

Nice!

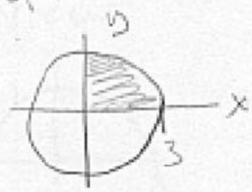
$y=x$ = θ value

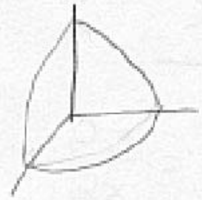
$\frac{\pi}{4}$

if it were whole circle $\theta = 0 \to 2\pi$

3. **Set up** iterated integrals for $\bar{z}$, the z coordinate of the center of mass of the first-octant portion of a sphere with radius 3 and uniform density k.

Top View





$z = \rho\cos\phi$

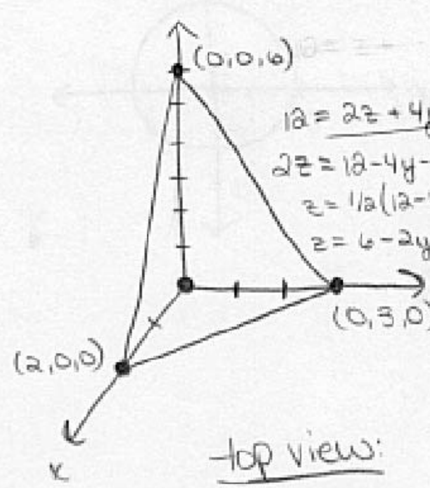
$$\bar{z} = \frac{\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{3} k\cdot\rho\cos\phi\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi}{\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{3} k\rho^2\sin\phi\,d\rho\,d\theta\,d\phi}$$

Excellent

4. **Set up** an iterated integral for the volume of the tetrahedron with vertices (0,0,0), (2,0,0), (0,3,0), and (0,0,6).

$$\int_0^{2}\int_0^{-3/2x+3}\int_0^{6-2y-3x} 1 \,dz\,dy\,dx$$

Great



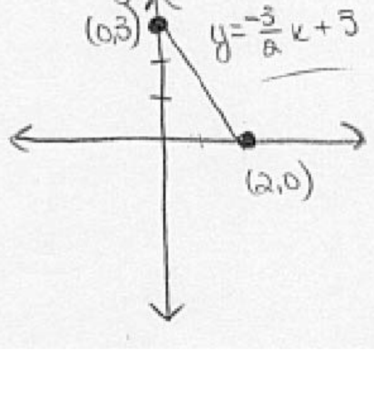
$12 = 2z + 4y + 6x$

$2z = 12 - 4y - 6x$

$z = 1/2(12 - 4y - 6x)$

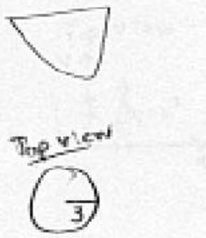
$z = 6 - 2y - 3x$

top view:



$y = \frac{-3}{2}x + 3$

5. **Set up** an iterated integral for the **surface area** of the portion of the paraboloid $z = x^2 + y^2$ below the plane $z = 9$.



$$SA = \iint_{R_1} \sqrt{1 + (f_x(x,y))^2 + (f_y(x,y))^2}\, dx\, dy$$

$$= \iint_{R_1} \sqrt{1 + (2x)^2 + (2y)^2}\, dx\, dy$$

$$= \iint_{R_1} \sqrt{1 + 4(x^2+y^2)}\, dx\, dy$$

$$= \iint_{R_2} \sqrt{1+4r^2}\; r\; dr\, d\theta$$

Perfect

$$\boxed{SA = \int_0^{2\pi} \int_0^3 \left[r\sqrt{1+4r^2} \right] dr\, d\theta}$$

7. **Compute the Jacobian** of the transformation $x = \frac{1}{3}(u+v)$, $y = \frac{1}{3}(v - 2u)$.

$x = 1/3\, u + 1/3\, v \qquad y = 1/3\, v - \frac{2}{3} u$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1/3 & -2/3 \\ 1/3 & 1/3 \end{vmatrix} \quad \left(\tfrac{1}{3}\right)\left(\tfrac{1}{3}\right) - \left(\tfrac{1}{3}\right)\left(-2/3\right) = \boxed{\tfrac{1}{3}}$$

Great

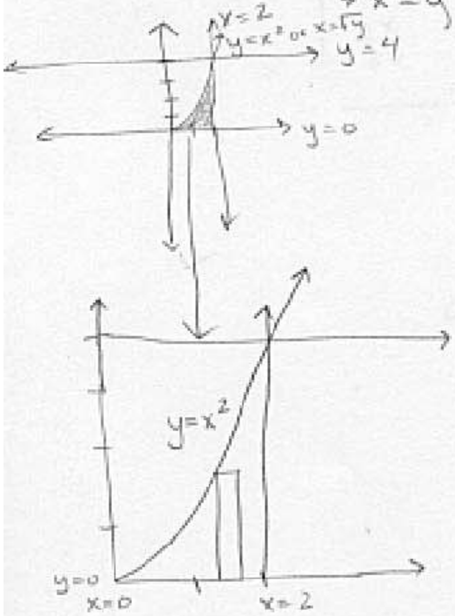
6. Evaluate $\int_0^4 \int_{\sqrt{y}}^2 \sqrt{4+x^3}\,dx\,dy$ exactly.

$y=4 \quad x=2$

$$\int_0^4 \int_{\sqrt{y}}^2 \sqrt{4+x^3}\,dx\,dy$$

$y=0 \quad x=\sqrt{y}$

$\hookrightarrow x^2=y$



$$\int_0^2 \int_0^{x^2} \sqrt{4+x^3}\,dy\,dx$$

$$\int_0^2 \left(y\sqrt{4+x^3} \right]_0^{x^2} dx$$

$$\int_0^2 \left(x^2\sqrt{4+x^3} - 0\sqrt{4+x^3}\right) dx$$

$$\int_0^2 \left(x^2\sqrt{4+x^3}\right) dx$$

$u = 4+x^3$

$du = 3x^2\,dx$

$\frac{du}{3} = x^2\,dx$

$$\int_0^2 \left(\sqrt{4+x^3}\,(x^2)\right) dx$$

$$\int_0^2 (\sqrt{u})\frac{du}{3}$$

Excellent

$$\tfrac{1}{3}\int_0^2 u^{1/2}\,du$$

$$\tfrac{1}{3}\left(\tfrac{2}{3}u^{3/2}\right]_{x=0}^{x=2}$$

$$\tfrac{2}{9}\left((4+x^3)^{3/2}\right]_0^2$$

$$\tfrac{2}{9}\left((4+2^3)^{3/2} - (4+0)^{3/2}\right)$$

$$\tfrac{2}{9}\left((4+8)^{3/2} - (4)^{3/2}\right)$$

$$\tfrac{2}{9}\left((12)^{3/2} - 2^3\right)$$

$$\tfrac{2}{9}\left(12^{3/2} - 8\right)$$

$$\tfrac{2}{9}\left(\sqrt{12}^{\,3} - 8\right)$$

$$\tfrac{2}{9}\left((2\sqrt{3})^3\right) - \tfrac{16}{9}$$

$$\tfrac{2}{9}\left(8(3)^{3/2}\right) - \tfrac{16}{9}$$

$$\tfrac{16}{9}(3)^{3/2} - \tfrac{16}{9}$$

$$\tfrac{16}{9}\left((3)^{3/2} - 1\right)$$

8. Biff is having some trouble with iterated integrals. Biff says "Man, we had this quiz and I know I did it wrong, 'cause I worked out this double integral and got zero. Volume can't be zero, so I must have screwed up, but I went over it twenty times and I have no idea what was wrong. It wasn't that complicated, either, the thing we integrated was just x, so I don't know how I messed up."

Explain clearly to Biff whether zero is automatically a wrong answer for a double integral where the integrand is x, and why.

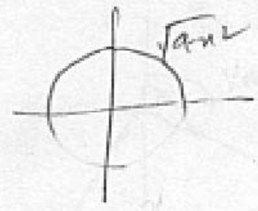
Zero is not necessorily a wrong answer for an integral with an integrand of X, becouse if the integrand is X, you may not be computing a volume. Infact, if you are computing an integral with an integrand of x, it is quite likely that you are finding the moment about the y axes (M_y, the top part for $\bar{x}$). There are varius instances where this will be $\emptyset$. For instance you may be finding M_y for a disk, centered at the origin with radius 1.

If the disk has a uniform density - it is easy to see that it will balance at the center, i.e. $M_y = \emptyset$. Thus this is an instance where a double integral with an integrand of x is zero.

Excellent!

9. Evaluate $\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{\sqrt{9-x^2-y^2}} 2dzdydx$ **exactly.**

$$= 2\left[\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{\sqrt{9-x^2-y^2}} dz\,dy\,dx\right]$$



This is the volume of 2 octants of a sphere with radius 3

$\therefore$ The integral equals $= 2\times\frac{1}{4}\times$ (vol. of sphere with $r=3$))

$$= \frac{1}{2}\times 2\frac{4}{\cancel{3}}\times 3^{\cancel{3}\,2}\pi = 2\times 9 = \boxed{18\pi}$$

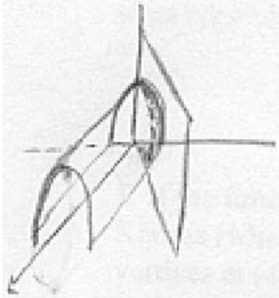
Nice Job

$z = \sqrt{9-x^2-y^2}$

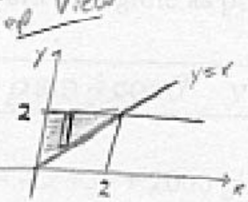
$\Rightarrow x^2+y^2+z^2 = 9$

radius $= 3$.

10. **Set up** an iterated integral to find the volume of the solid in the first octant bounded by the elliptic cylinder $y^2 + 4z^2 = 4$ and the plane $y = x$.



Top View



$y^2 + 4z^2 = 4$

$4z^2 = 4 - y^2$

$z^2 = 1 - \frac{y^2}{4}$

$z = \pm\sqrt{1-\frac{y^2}{4}}$

Well done!

$$\int_0^2\int_x^2 \sqrt{1-\frac{y^2}{4}}\,.dy\,dx$$