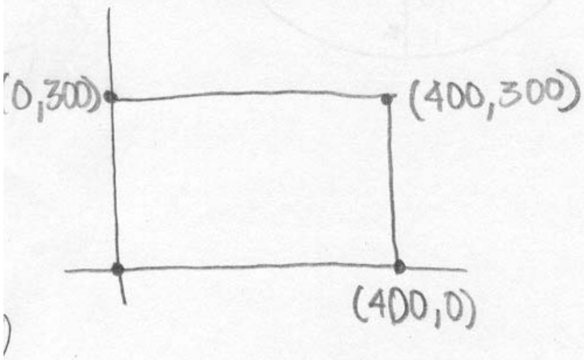
Thanks! :) **Exam 2** **Calc 3** **10/29/2004**

Each problem is worth 10 points. For full credit provide complete justification for your answers.

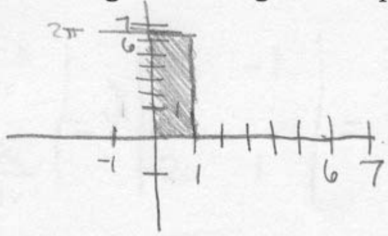
1. Jon has done intensive research on the vulture population in the state of Wyoming. It turns out that if you regard Wyoming as a rectangle, with its lower left corner at the origin, lower right corner at the point (400, 0), upper left corner at the point (0, 300), and upper right corner at the point (400, 300), then the vulture population density is roughly given by the function $v(x, y) = 300 + 1.5x - 0.5y$ vultures per square mile. Write an iterated integral for the total vulture population in Wyoming.



$$\int_{x=0}^{x=400} \int_{y=0}^{y=300} (300+1.5x-0.5y)\,dy\,dx$$

Good

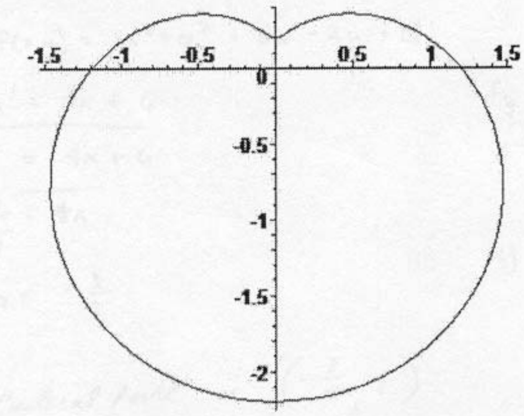
2. Sketch the region of integration represented by the integral $\int_0^{2\pi}\int_0^1 (x^2+y^2)\,dx\,dy$.



It is in $d\underline{x}\,d\underline{y}$ so it is a rectangle.
If it was in polar or $dr\,d\theta$ it would be a circle.

Yep!

3. Set up an iterated integral for the area of the region bounded by the curve with polar equation $r = 1.2 - \sin\theta$, shown below.



Polar Coordinates

$$\boxed{\int_0^{2\pi} \int_0^{1.2-\sin\theta} 1 \; r \, dr \, d\theta}$$

Great

4. Jon plans to try to popularize a new coordinate system, which he's calling "Jon's Parabolic Coordinates". It involves the transformations $x = u^2 - v$ and $y = v + u^2$. Find the Jacobian for this transformation.

$x = u^2 - v \quad y = v + u^2$

$$\begin{vmatrix} \partial x/\partial u & \partial y/\partial u \\ \partial x/\partial v & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} 2u & 2u \\ -1 & 1 \end{vmatrix} = 2u - (-2u)$$

$$= 2u + 2u$$

Excellent

$$= 4u \, du dv$$

5. Find all critical points of the function $f(x, y) = 2x^2 + y^2 + 6x - 2y + 17$ and classify them as maxima, minima, or saddle points.

$$f(x,y) = 2x^2 + y^2 + 6x - 2y + 17$$

$$f_x(x,y) = 4x + 6$$

$$f_y(x,y) = 2y - 2$$

Now,

$4x + 6 = 0$ $2y - 2 = 0$

or, $x = \frac{-3}{2}$ or, $y = 1$

$\therefore$ Critical point is $\left(-\frac{3}{2}, 1\right)$

Now,

$f_{xx}(x,y) = 4$ $f_{xy}(x,y) = 0$

$f_{yy}(x,y) = 2$

$$\therefore D(x,y) = f_{xx}(x,y) \cdot f_{yy}(x,y) - [f_{xy}(x,y)]^2$$

$$D(-3/2, 1) = 4 \cdot 2 - 0 = 8$$

Excellent

Since $D > 0$ the point $(-3/2, 1)$ should be a max or min.

But $f_{xx} > 0$ i.e. it is concave up.

$\therefore (-3/2, 1)$ is a minima.

6. Find the extreme values of $f(x, y) = x + y^2$ subject to the constraint $x^2 + y^2 = 4$.

$$\text{grad } f(x,y) = \lambda \cdot \text{grad } g(x,y)$$

$$g(x,y) = k$$

$$\text{grad } f(x,y) = \vec{i} + 2y\vec{j}$$

$$\text{grad } g(x,y) = 2x\vec{i} + 2y\vec{j}$$

$$\therefore (\vec{i} + 2y\vec{j}) = \lambda\,(2x\vec{i} + 2y\vec{j})$$

$$\therefore \quad 1 = \lambda \cdot 2x$$

$$2y = \lambda\, 2y$$

Now, $2y(1-\lambda) = 0$

Beautiful Work.

Either $y = 0$ or $\lambda = 1$

If $y = 0$

$x^2 = 4$

or, $x = \pm 2$

If $\lambda = 1$

then $x = \frac{1}{2}$

So, $\frac{1}{4} + y^2 = 4$

or, $1 + 4y^2 = 16$

or, $4y^2 = 15$

$\therefore y = \pm \frac{\sqrt{15}}{2}$

$\therefore$ The critical points are $(2,0), (-2,0), \left(\frac{1}{2}, \frac{\sqrt{15}}{2}\right), \left(\frac{1}{2}, -\frac{\sqrt{15}}{2}\right)$

Now, $f(2,0) = 2$

$f(-2,0) = -2$

$f\left(\frac{1}{2}, \frac{\sqrt{15}}{2}\right) = \frac{17}{4}$

$f\left(\frac{1}{2}, -\frac{\sqrt{15}}{2}\right) = \frac{17}{4}$

$\therefore$ Max is $\frac{17}{4}$ & min is -2

7. Bunny is a calculus student at a large state university, and she's having some trouble. Bunny says "Wow, this stuff is hard. We have to do this stuff in these polar coordinations, and I totally don't get it. Some of the people in the class saw it before I guess, so they get it and it's totally unfair. I mean, I get trig okay so I understand the theta and everything, but I just don't understand where the axis thingies are. I thought the professor said the thetas were where the x axis is, so then the r would be where the y axis is, right? So is it just that you use different letters, and that's all? I think maybe I'm really lost."

Explain polar coordinates in terms Bunny can understand.

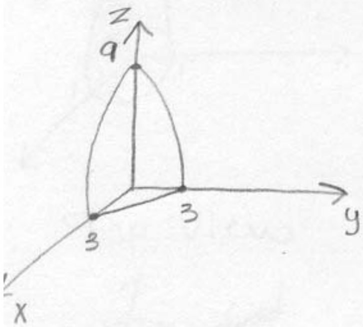
Sorry Bunny, you can't just replace x and y with r and θ. In cartesian coordinates (x,y), x and y both measure distance in a certain direction... good they are always ~~up~~ the same directions ~~in the polar coords~~ relative to one-another.

In polar coordinates though, r measures distance in any direction (well, any direction on a plane...). R is still a distance, like x and y, but its direction is determined by θ — θ is an angle, not a direction/distance.

Excellent answer.

For example, the point $(1, \pi/4)$ in polar coordinates is a distance of 1 unit from the origin, and in the direction of $\pi/4$ (radians measured counter-clockwise) from what is the positive-x axis in cartesian coordinates.

8. Jon plans to earn a little spare cash on the side by starting a mathematical knick-knack business. His first product will be mathematical bookends, which will be shaped like the solid bounded by the surface $z = 9 - (x + y)^2$ and the planes $x = 0$, $y = 0$, and $z = 0$ (where all coordinates are measured in inches). If the density of the concrete used to form the bookends is approximately 0.8 ounces per cubic inch, set up an iterated integral for the total weight in ounces of one bookend.



$z = 9 - (x+y)^2$

z-intercept: $z = 9$

y-intercept: $0 = 9 - y^2$

$y^2 = 9$

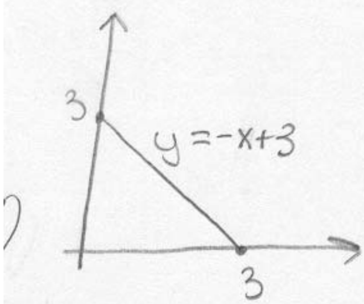
$y = 3$

x-intercept: $0 = 9 - x^2$

$x^2 = 9$

$x = 3$

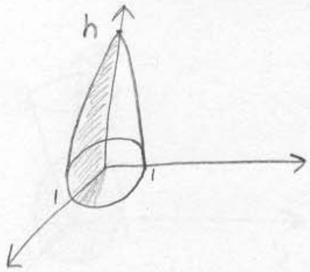
Top View



$$\int_0^3 \int_0^{-x+3} \int_0^{9-(x+y)^2} .8 \, dz \, dy \, dx$$

Well done

9. Compute the exact value of the integral $\int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\left(h-\sqrt{x^2+y^2}\right)dy\,dx$.



$r^2 = x^2+y^2$

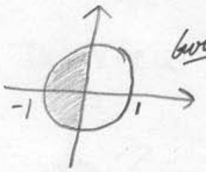
$r = \sqrt{x^2+y^2}$

h = constant, doesn't convert

switch to polar coordinates: Yes!

$$\int_{\pi/2}^{3\pi/2}\int_0^1 (h-r)\,r\,dr\,d\theta$$

Top View



Good

$$\int_{\pi/2}^{3\pi/2}\int_0^1 (hr-r^2)\,dr\,d\theta$$

$$\int_{\pi/2}^{3\pi/2}\left(\tfrac{1}{2}hr^2-\tfrac{1}{3}r^3\right)\Big|_0^1\,d\theta$$

$$\int_{\pi/2}^{3\pi/2}\left[\tfrac{1}{2}h(1)^2-\tfrac{1}{3}(1)^3\right]-\left[\cancel{\tfrac{1}{2}h(0)^2-\tfrac{1}{3}(0)^3}\right]d\theta$$

$$\int_{\pi/2}^{3\pi/2}\left(\tfrac{1}{2}h-\tfrac{1}{3}\right)d\theta$$

$$\tfrac{1}{2}h\theta-\tfrac{1}{3}\theta\Big|_{\pi/2}^{3\pi/2}$$

Great Job!

$$\left[\tfrac{1}{2}h\left(\tfrac{3\pi}{2}\right)-\tfrac{1}{3}\left(\tfrac{3\pi}{2}\right)\right]-\left[\tfrac{1}{2}h\left(\tfrac{\pi}{2}\right)-\tfrac{1}{3}\left(\tfrac{\pi}{2}\right)\right]$$

$$\left(\tfrac{3\pi}{4}h-\tfrac{\pi}{2}\right)-\left(\tfrac{\pi}{4}h-\tfrac{\pi}{6}\right)$$

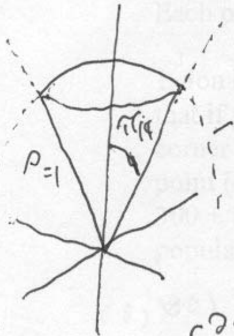
$$\tfrac{3\pi}{4}h-\tfrac{\pi}{2}-\tfrac{\pi}{4}h+\tfrac{\pi}{6}$$

$$\tfrac{2\pi}{4}h-\tfrac{\pi}{2}+\tfrac{\pi}{6}$$

$$\tfrac{\pi}{2}h-\tfrac{\pi}{2}+\tfrac{\pi}{6} = \tfrac{\pi}{2}h-\tfrac{\pi}{3}$$

Cone facing up from origin

10. Set up an iterated integral for the volume bounded below by the surface $z^2 = x^2 + y^2$ and above by the surface $x^2 + y^2 + z^2 = 1$ and evaluate it to find the volume of this solid.



$$\boxed{x=\rho\sin\phi\cos\theta \quad y=\rho\sin\phi\sin\theta \quad z=\rho\cos\phi \quad J=\rho^2\sin\phi}$$

$$\int_0^{2\pi}\int_0^{\pi/4}\int_0^1 \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/4}\left[\frac{\rho^3}{3}\sin\phi\Big|_0^1\right] d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi/4}\frac{\sin\phi}{3}\, d\phi\, d\theta$$

$$= \frac{1}{3}\int_0^{2\pi}\left[-\cos\phi\Big|_0^{\pi/4}\right] d\theta = \frac{1}{3}\int_0^{2\pi}\left[-\cos\frac{\pi}{4} - (-\cos(0))\right] d\theta$$

10

$$= \frac{1}{3}\int_0^{2\pi} 1 - \frac{\sqrt{2}}{2}\, d\theta = \frac{1}{3}\left[\theta - \frac{\sqrt{2}}{2}\theta\right]_0^{2\pi} = \frac{1}{3}\left(2\pi - \frac{\sqrt{2}}{2}\cdot 2\pi - 0 - 0\right)$$

$$= \boxed{\frac{2\pi}{3} - \frac{\sqrt{2}\pi}{3}}$$

Nice Work.