**Exam 2 Real Analysis 1 11/12/2004**

Each problem is worth 10 points. Show adequate justification for full credit. Don't panic.

1. State the definition of the derivative of f at $x = a$.

Let $f: D \to \mathbb{R}$ & $D \subseteq \mathbb{R}$. Define the derivative of f at x=a by the following:

$$f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}, \text{ (as long as the limit is a finite \#!)}$$

Great

2. State the Intermediate Value Theorem.

If f is continuous on $[a,b]$ and K is between F(a) and F(b) then there exists a $c \in (a,b)$ such that $F(c)=K$

Good

3. State and prove the Difference Rule for derivatives.

If f and g are differentiable then $(f-g)'(x) = f'(x) - g'(x)$.

Proof: Well, $(f-g)'(x) = \lim_{x \to a} \frac{(f-g)(x) - (f-g)(a)}{x-a}$

$$= \lim_{x \to a} \frac{f(x) - g(x) - f(a) + g(a)}{x-a}$$

$$= \lim_{x \to a} \frac{f(x) - f(a)}{x-a} + \frac{g(a) - g(x)}{x-a}$$

$$= \lim_{x \to a} \frac{f(x) - f(a)}{x-a} - \lim_{x \to a} \frac{g(x) - g(a)}{x-a}$$

Excellent

$$= f'(x) - g'(x)$$, since we know these derivatives exist. □

4. Give an example of a function which is differentiable and continuous on $(a, b)$ but which does not satisfy the conclusion of the Mean Value Theorem.

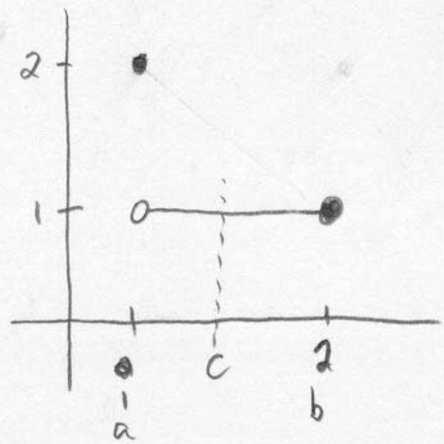
If cont. $[a,b]$, diff on $(a,b)$ $\Rightarrow \exists\, c \in (a,b) \ni f'(c) = \dfrac{f(b)-f(a)}{b-a}$

$$\boxed{\frac{f(b)-f(a)}{b-a} = \frac{1-2}{2-1} = -1}$$

$f'(x)\ \forall\, x \in (a,b) = 0.$

$-1 \neq 0.$

Cont. & Diff. on $(a,b)$




Excellent

5. Prove that if f($x$) is an even function defined on $\mathbb{R}$, then f′($x$) is an odd function.

$f(-x) = f(x)$ even function.

Taking Derivative: $-f'(-x) = f'(x)$ by Chain Rule Nice

Mult. by $(-1)$: $f'(-x) = -f'(x)$

which means that $f'(x)$ is an odd function.

6. State and prove the Squeeze Theorem for functions f, g, and h.

If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} h(x) = L$, and $f(x) \le g(x) \le h(x)$ for all $x \in D$, then $\lim_{x\to a} g(x) = L$.

Proof: First note that $a$ must be an accumulation point of $D$, since $\lim_{x\to a} f(x)$ and $\lim_{x\to a} h(x)$ are defined and finite. Yes!

We know from our hypotheses that given an $\varepsilon > 0$, $\exists\ \delta_1$ such that $|f(x)-L| < \varepsilon$ for $0 < |x-a| < \delta_1$ and $x \in D$; also, given an $\varepsilon > 0$, $\exists\ \delta_2$ such that $|h(x)-L| < \varepsilon$ for $0 < |x-a| < \delta_2$ and $x \in D$. We can expand our inequalities so that we have $-\varepsilon < f(x)-L < \varepsilon$ and $-\varepsilon < h(x)-L < \varepsilon$.

Now, our other hypothesis says that $f(x) \le g(x) \le h(x)$, so we can say $f(x)-L \le g(x)-L \le h(x)-L$. Thus, by transitivity, $-\varepsilon \le g(x)-L \le \varepsilon$, or $|g(x)-L| < \varepsilon$.

So we know that for any $\varepsilon > 0$, $|g(x)-L| < \varepsilon$ for $0 < |x-a| < \min\{\delta_1, \delta_2\}$ and $x \in D$. $\square$

Wonderful!

7. Prove that if a function f is differentiable at $x = a$, then f is continuous at $x = a$.

Prop: (Differentiability Implies continuity). If $f: D \to \mathbb{R}$, and $f'(a)$ exists, then f is continuous at $x=a$.

Proof: well, let's show that $\lim_{x \to a} [f(x) - f(a)] = 0$, and know that this suffices. But, $\lim_{x \to a} [f(x) - f(a)] =$

$$\lim_{x \to a} \left[ \frac{f(x) - f(a)}{x - a} \cdot (x-a) \right] = f'(a) \cdot 0 = 0,$$ since

$f'(a)$ is finite. □

Nice!

8. State and Prove Rolle's Theorem.

If $f$ is continuous on $[a,b]$, differentiable on $(a,b)$, and $f(a)=f(b)$, then $\exists\ c \in (a,b)$ such that $f'(c)=0$.

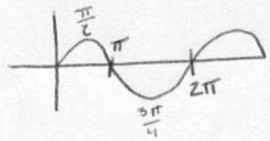
Proof:

① Well, if $f$ is a constant function on $[a,b]$, say $f(x)=M$, then $f'(x)=0 \quad \forall\ x \in [a,b]$.

② If $f(x) > f(a)$ for some $x \in [a,b]$: There must be some maximum, call it $k$. Then by the Extreme Value Theorem, $f$ must attain its max on $[a,b]$, since $f$ is continuous on $[a,b]$; i.e. $\exists\ c \in [a,b] \ni f(c)=k$. We know that $c \neq a$ and $c \neq b$ because there is some $f(x) > f(a) = f(b)$. So we have $c \in (a,b)$ and $f(c)=k$, the max. of $f$. Then by Fermat's Theorem, $f'(c)=0$.

③ If $f(x) < f(a)$ for some $x \in [a,b]$: There must be some minimum, call it $n$. Since $f$ is continuous on $[a,b]$, the Extreme Value Theorem says $f$ must attain its minimum on $[a,b]$; i.e. $\exists\ d \in [a,b] \ni f(d)=n$. By similar reasoning (as above, in case 2), we know $d \neq a$ and $d \neq b$, so $d \in (a,b)$. By Fermat's Theorem, then, $f'(d)=0$. $\square$

Absolutely Perfect!




9. Does $\lim_{x\to\infty} \sin\sqrt{x}$ exist?

No, still oscillates.

Pick two sequences.

$\{x_n\} = (n\pi)^2$

$\{t_n\} = (2n\pi + \frac{\pi}{2})^2$

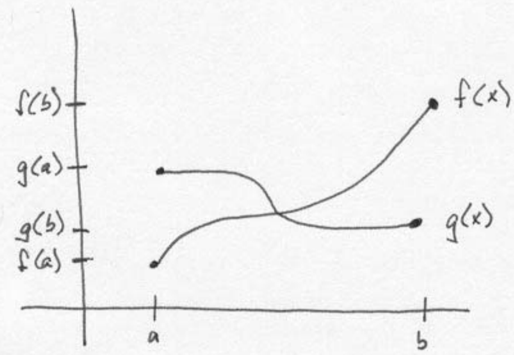
$f(\{x_n\}) = \sin\sqrt{(n\pi)^2} = \sin n\pi = 0$

$f(\{t_n\}) = \sin\sqrt{(2n\pi + \frac{\pi}{2})^2} = \sin 2n\pi + \frac{\pi}{2} = 1$

Nicely done!

so the function converges to 0 and 1, respectively and since you can't have more than one limit because limits are unique, then the limit does not exist. $\square$

10. Suppose that f and g are differentiable functions defined on $\mathbb{R}$ and that for some real numbers $a$ and $b$ (with $a < b$) we have $f(a) < g(a)$ and $f(b) > g(b)$. Does there have to exist a $c \in (a,b)$ for which $f(c) = g(c)$?




Yes, there does!

Proof: Well, first notice that f and g are necessarily continuous on $[a, b]$ (since they're differentiable on $\mathbb{R}$) and differentiable on $(a, b)$ (for the same reason). Also notice that $f(a) < g(a) \Rightarrow f(a) - g(a) < 0$, or $(f-g)(a) < 0$, while $f(b) > g(b) \Rightarrow f(b) - g(b) > 0$ or $(f-g)(b) > 0$. But $(f-g)(x)$ is continuous on $[a, b]$ and differentiable on $(a, b)$ as a difference of continuous and differentiable functions, respectively, so the hypotheses of the Intermediate Value Theorem apply. Then since $0$ is between $(f-g)(a)$ and $(f-g)(b)$, by I.V.T. there must exist a $c \in (a, b)$ for which $(f-g)(c) = 0$, and thus $f(c) = g(c)$, as desired. $\square$