Problem Set 4

Prop:

) If $\{a_n\} \to 0$ then $\{|a_n|\} \to 0$.

Proof: Well, we know that for any $\varepsilon > 0$, there exists an n^* such that

$$n > n^* \Rightarrow |a_n - 0| < \varepsilon$$

$$\text{or } |a_n| < \varepsilon.$$

Little lemma: $||x|| \stackrel{?}{=} |x|$.

$|x| > 0$ by definition of absolute value.

so $||x|| = |x|$ by definition of absolute value and because $|x| > 0$

Using our lemma, we can say that $|a_n| = ||a_n|| = ||a_n| - 0|$.

So, since $|a_n| < \varepsilon$,

$||a_n| - 0| < \varepsilon$, as desired.

Thus, for a given $\varepsilon > 0$, $\exists\ n^* \ni n > n^* \Longrightarrow ||a_n| - 0| < \varepsilon$,

so $\{|a_n|\}$ converges to 0. $\square$

Beautiful!

) Prop: If $\{|a_n|\} \to 0$, then $\{a_n\} \to 0$.

Proof: We know that, for any $\varepsilon > 0$, $\exists\ n^* \ni n > n^* \implies ||a_n| - 0| < \varepsilon$

or $||a_n|| < \varepsilon$.

Lemma: (same as for #1):

$||a_n|| \stackrel{?}{=} |a_n|$

$|a_n| > 0$ by definition of absolute value,

so $||a_n|| = |a_n|$ by def. of abs. value and $|a_n| > 0$.

So we can say $||a_n|| = |a_n| = |a_n - 0|$

and since $||a_n|| < \varepsilon$,

$|a_n - 0| < \varepsilon$, as desired.

So, for a given $\varepsilon > 0$, $\exists\ n^* \ni n > n^* \implies |a_n - 0| < \varepsilon$.

Thus, $\{a_n\}$ converges to 0. $\square$

Great job.

3.) Prop: If the sequence $\{a_n\}$ converges to A, then the sequence $\{|a_n|\}$ converges to $|A|$.

Proof: Well, saying that $\{a_n\}$ converges to A means that

$$\forall \varepsilon > 0 \ \exists n^* \in \mathbb{N} \ni n > n^* \Rightarrow |a_n - A| < \varepsilon. \ \forall n \in \mathbb{N}.$$

$$|a_n - A| < \varepsilon.$$

Well, if $a_n, A \geq 0$, then $|a_n| = a_n$ and $|A| = A$, so

$$||a_n| - |A|| = |a_n - A| < \varepsilon.$$

so $||a_n| - |A|| < \varepsilon$ ✓

But if $a_n, A < 0$, then $|a_n| = -a_n$ and $|A| = -A$, so

$$|-|a_n| - -|A|| = |a_n - A| < \varepsilon$$

$$|-(|a_n| - |A|)| < \varepsilon$$

if $|a_n| > |A|$ then $-(|a_n| - |A|) < 0$

so $|-(|a_n| - |A|)| = -(-(|a_n| - |A|))$

$= (|a_n| - |A|) = ||a_n| - |A||$

so $|-(|a_n| - |A|)| < \varepsilon$

$||a_n| - |A|| < \varepsilon$. ✓

all steps by def of abs. value or transitivity

if $|a_n| < |A|$, then $-(|a_n| - |A|) \geq 0$, so $|a_n| - |A| < 0$

$|-(|a_n| - |A|)| = -(|a_n| - |A|) = ||a_n| - |A||$

$||a_n| - |A|| = |-(|a_n| - |A|)| < \varepsilon$

$||a_n| - |A|| < \varepsilon$. ✓

We don't need to consider if $a_n > 0$ and $A < 0$, or $a_n < 0$ and $A > 0$ because we know that $\{a_n\} \to A$, so they are either both ≥ 0 or < 0. Unless A is 0 but we proved this holds for 0 in problem #1.

Outstanding.

Prop: If $\{|a_n|\} \to |A| \Rightarrow \{a_n\} \Rightarrow A$.

False.

counter example:

let $\varepsilon > 0$ be given.

$a_n = (-1)^n$

$|a_n| = 1$ for all n

$\Rightarrow |a_n|$ converges to 1

$|a_n - 1| < \varepsilon$ for all n

$||1-1|| = |0| = 0 < \varepsilon$

But, for a_n to converge to 1

$|a_n - 1| < \varepsilon$ for all $n > n^*$

But, a_n alternates from -1 to 1 and

$|-1-1| = |-2| = 2 \nless \varepsilon$

$\Rightarrow a_n$ does not converge to A. $\square$

Great

Prop: If the sequence $\{a_n\}$ converges to 0, and the sequence b_n is bounded, then the sequence $\{a_n b_n\}$ converges to 0.

Proof: Well, let some $\varepsilon > 0$ be given. Then there exists an $n_1 \in \mathbb{N}$ so that for $n > n_1$ we have $|a_n - 0| < \frac{\varepsilon}{K}$, for some $K > 0$. We also know that b_n is bounded so there exists some $M \in \mathbb{N}$ such that for any $n \in \mathbb{N}$, $|b_n| < M$. Heck, lets let $K = M$, so we have $|a_n - 0| < \frac{\varepsilon}{M}$ and $|b_n| < M$, but by lemma 2 we have $|a_n - 0| \cdot |b_n| < \frac{\varepsilon}{\cancel{M}} \cdot \cancel{M}$ or by theorem 1.8.5.(d) $|(a_n - 0) \cdot b_n| < \varepsilon$ or $|(a_n \cdot b_n) - 0| < \varepsilon$. So, by the definition of convergence $\{a_n b_n\} \to 0$. $\square$

lemma 2: For this we need to prove if $a < b$ and $c < d$ then $ac < bd$ for only 1 case since $|a_n - 0| \geq 0$, $\frac{\varepsilon}{M} > 0$, $M \in \mathbb{N} \Rightarrow M > 0$ and $|b_n| \geq 0$. So, let $a < b$ and $c < d$, where $a \geq 0$, $b > 0$, $c \geq 0$, $d > 0$ as above. Then by axiom 1a we have $ac < bc$ and $bc < bd$, using transitivity we have $ac < bd$. Yes.

e) Prop: If $\{a_n\} \to 0$ and $\{b_n\}$ is some other sequence,

then $\{a_n b_n\} \to 0$.

Counterexample: Let $\{a_n\} = \{\frac{1}{n}\}$ and $\{b_n\} = \{n\}$.

Then $\{a_n\} = \{\frac{1}{n}\} \to 0$.

(prove): Let $n^* < \frac{1}{\varepsilon}$. Then $n > n^*$ implies $n < \frac{1}{\varepsilon}$, or $\frac{1}{n} < \varepsilon$.

So $|\frac{1}{n} - 0| < \varepsilon$, as desired.

$\{b_n\}$ diverges.

These satisfy our "if" conditions.

Now: $\{a_n b_n\} \stackrel{?}{\to} 0$.

We'll show that $\{a_n b_n\}$ actually converges to 1:

Given $\varepsilon > 0$, n^* can be any natural number.

Then $0 < \varepsilon$, or $|1-1| < \varepsilon$.

So $|(\frac{1}{n})(n) - 1| < \varepsilon$, or $|a_n b_n - 1| < \varepsilon$, as desired.

Theorem 2.1.9 says that limits are unique, so $\{a_n b_n\}$ cannot converge to 0. $\Rightarrow\!\Leftarrow$. $\square$

Nice