Problem Set 5

1) Prop: The sequence $\left\{\frac{3}{\sqrt{n}+2}\right\}$ converges to 0.

Proof: Let $\varepsilon > 0$ be given. Choose $n^* \geq \frac{(3-2\varepsilon)^2}{\varepsilon^2}$.

Then for $n > n^*$:

$$n > \frac{(3-2\varepsilon)^2}{\varepsilon^2}$$

$$\varepsilon^2 n > (3-2\varepsilon)^2$$

$$(\varepsilon\sqrt{n})^2 > (3-2\varepsilon)^2$$

$$\varepsilon\sqrt{n} > 3-2\varepsilon$$

$$\varepsilon\sqrt{n} + 2\varepsilon > 3$$

$$\varepsilon(\sqrt{n}+2) > 3$$

$$\varepsilon > \frac{3}{\sqrt{n}+2} \quad \text{or} \quad \frac{3}{\sqrt{n}+2} < \varepsilon.$$

But $\frac{3}{\sqrt{n}+2} = \left|\frac{3}{\sqrt{n}+2}\right|$ because $\sqrt{n} > 0$, so $\frac{3}{\sqrt{n}+2} > 0$.

So $\frac{3}{\sqrt{n}+2} = \left|\frac{3}{\sqrt{n}+2} - 0\right| < \varepsilon$, as desired. $\square$

Wonderful

2) *Prop*: If $\{a_n\}$ diverges to $+\infty$ and $K \in \mathbb{R}$, then $\{a_n - K\}$ diverges to $+\infty$.

Proof: Since $\{a_n\}$ diverges to $+\infty$, then for any $M \in \mathbb{N}$, there exists n^* such that $a_n > M$ for all $n > n_1^*$.

If $K < 0$, then $a_n - K > a_n > M$

so $a_n - K > M$ for all $n > n_1^*$.

If $K \geq 0$: Since $\{a_n\}$ diverges to $+\infty$, then for an arbitrary real number, $M+K$, there exists an n_2^* such that

$$a_n > M + K \text{ for all } n > n_2^*$$

so $a_n - K > M + K - K$

or $a_n - K > M$ for all $n > n_2^*$.

Thus $\{a_n - K\}$ diverges to $+\infty$. $\square$

5

Excellent

3) Prop: If $\{a_n\}$ is increasing and $K \in \mathbb{R}$ then $\{K \cdot a_n\}$ is monotone.

Proof: Let $n, m \in \mathbb{N}$ and $n > m$. Then $a_n \geq a_m$ by definition of increasing.

For $K \geq 0$: $a_n \geq a_m$

$K \cdot a_n \geq K \cdot a_m$ by axiom.

So if $K \geq 0$, $\{K \cdot a_n\}$ is increasing.

For $K < 0$: $a_n \geq a_m$

$K \cdot a_n \leq K \cdot a_m$ by in-class proof.

So if $K < 0$, $\{K \cdot a_n\}$ is decreasing.

Since $\{K \cdot a_n\}$ is always either increasing or decreasing, it is monotone. $\square$

Perfect

4.) Show that $\left\{\frac{n}{n+1}\right\}$ is increasing.

To be increasing it means that $\forall\, m,n \in \mathbb{N}$ if $m < n \Rightarrow a_m \leq a_n$.

Well, let $m,n \in \mathbb{N}$ and $m < n$.

We know that $m < n$

Add mn to both sides : $mn + m < mn + n$ axiom 11

factor both sides : $m(n+1) < n(m+1)$ axiom 8

divide both sides by $(n+1)$: $m < \frac{n(m+1)}{(n+1)}$ axiom 12

divide both sides by $(m+1)$: $\frac{m}{(m+1)} < \frac{n}{(n+1)}$ axiom 12

which says that if $m < n$ for all $m,n \in \mathbb{N}$ if $m < n$, then $\frac{m}{(m+1)} < \frac{n}{(n+1)}$, which is the definition of strictly increasing, which means not only is it increasing but it is strictly increasing, as desired. $\square$

Nice job. ✓