Problem Set 9

Happy Thanksgiving!

1) Can there exist a function f, interval [a, b], and partition P such that $L(P, f)$ is greater than the true value of the integral?

No... and here's why:

We'll use the fact that $L(P,f) \le \underline{\int_a^b} f$, which appears on p. 244 of the text, so let's prove that:

We know $\underline{\int_a^b} f = \sup\{L(Q,f) \mid Q \text{ a partition of } [a,b]\}$, meaning it is the greatest of all possible lower sums, for <u>any</u> Q.

So, since $L(P,f)$ is just another lower sum, we must have

$$L(P,f) \le \underline{\int_a^b} f.$$

Now, we also know that if $\underline{\int_a^b} f = \overline{\int_a^b} f$, then that is the value of the true integral, so $\underline{\int_a^b} f = \overline{\int_a^b} f = \int_a^b f$, by definition 6.1.6.

Thus we have $L(P,f) \le \int_a^b f$, which is what we needed to show. $\square$

Nice!

(a) No, since $U(P,f)$ is an upper sum that uses

$M_k(f) = \sup\{ f(n) \mid n \in [x_{k-1}, x_k] \}$ and that's greater than

5 $f(c_k) \in [x_{k-1}, x_k]$ by definition because M_k is a least upper bound.

So, $\sum_{k=1}^{n} M_k \Delta x_k \geq \sum_{k=1}^{n} f(c_k) \cdot \Delta x_k$

$\therefore\ S(P,f) \leq U(P,f)$ Yup.

③ Can there exist a function f, interval $[a,b]$, and partitions P and Q with Q having more subintervals than P, but with $L(P,f) < L(Q,f)$.

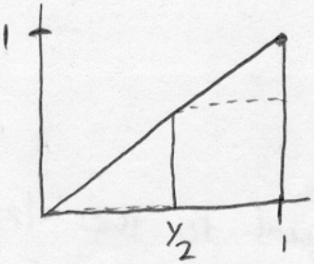
Yes.

For $f(x)=x$ from $[0,1]$

let $P = \{0, \frac{1}{2}, 1\}$

so, $m_1 = 0 \quad m_2 = \frac{1}{2}$

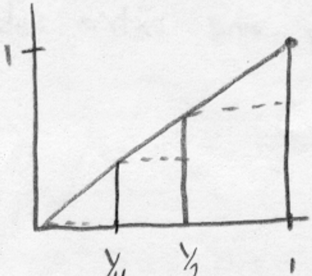
$$L(P,f) = 0 \cdot \frac{1-0}{2} + \frac{1}{2}\left(\frac{1-0}{2}\right) = \frac{1}{4}$$



Now take Q, a refinnent of P, to be $Q = \{0, \frac{1}{4}, \frac{1}{2}, 1\}$. Note this partition has one extra subinterval.

so, $m_1 = 0 \quad m_2 = \frac{1}{4} \quad m_3 = \frac{1}{2}$

$$L(Q,f) = 0 \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{2} \frac{1}{2} = \frac{1}{16} + \frac{1}{4} = \frac{5}{16}$$

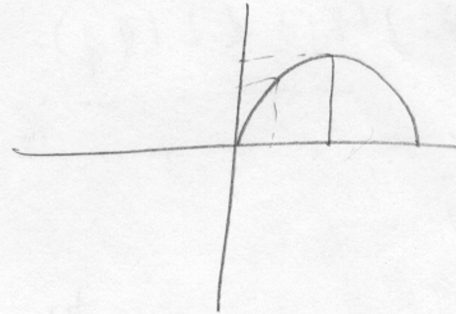


Great

Since $\frac{1}{4} < \frac{5}{16}$

$L(P,f) < L(Q,f)$. □

4. Taking $f(x) = \sin x$ in $[0, \pi]$



if we take $P = \{\frac{\pi}{2}, \pi\}$

then $R(P,f) = 1\times\frac{\pi}{2} + \frac{\pi}{2}\times 0$

$= \frac{\pi}{2} = 1.57$

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Let $P' = \{\frac{\pi}{4}, \frac{\pi}{2}, \pi\}$ Good

then $R(P',f) = \frac{1}{\sqrt{2}}\cdot\frac{\pi}{4} + 1\times\frac{\pi}{4} + \frac{\pi}{2}\times 0 = \frac{\pi}{4\sqrt{2}} + \frac{\pi}{4} = 1.34$

$$\therefore \boxed{\underset{=1.34}{R(P',f)} < \underset{=1.57}{R(P,f)} \le \int_0^{\pi} f = 2}$$