Exam 1   Calc 3   9/23/2005

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. State the formal definition of the partial derivative of a function f(*x*, *y*) with respect to *y*.

The partial derivative $= f_y(x,y) = \lim_{h \to 0} \frac{f(x,(y+h)) - f(x,y)}{h}$

Good

2. Compute the directional derivative of the function $g(x, y) = xy^3 - 2x$ at the point (2,3) in the direction of the vector $4\vec{i} - 3\vec{j}$.

$g_x(x,y) = y^3 - 2$ $g_x(2,3) = 25$

$g_y(x,y) = 3xy^2$ $g_y(2,3) = 54$

$\vec{v} = \langle 4, -3 \rangle$

$\|\vec{v}\| = 5$

$\vec{u} = \langle \frac{4}{5}, \frac{-3}{5} \rangle$

the vecter is now a unit vector

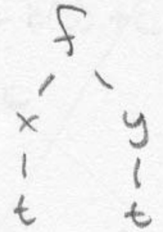
grad $\cdot\ \vec{u}$

$25 \cdot \frac{4}{5} + \frac{-3}{5} \cdot 54 =$

$20 - 32.4 = -12.4$

Well Done!

3. If f is a function of the two variables x and y, and x and y are in turn both functions of the variable t, write the appropriate version of the chain rule for $\frac{df}{dt}$. Make clear which, if any, of your derivatives are partials.



$$\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

partial derivatives

Yep!

4. Show that $\lim_{(x,y)\to(0,0)} \frac{x^2+2xy+y^2}{x^2+y^2}$ does not exist.

Approaching along $x=0$

$$\lim_{(0,y)\to(0,0)} \frac{x^2+2xy+y^2}{x^2+y^2} = \lim_{(0,y)\to(0,0)} \frac{y^2}{y^2} = \underline{1}$$

Approaching along $y=0$

$$\lim_{(x,0)\to(0,0)} \frac{x^2+2xy+y^2}{x^2+y^2} = \lim_{(x,0)\to(0,0)} \frac{x^2}{x^2} = \underline{1}$$

Approaching along $x=y$

$$\lim_{(y,y)\to(0,0)} \frac{x^2+2xy+y^2}{x^2+y^2} = \lim_{(y,y)\to(0,0)} \frac{y^2+y^2+2y^2}{y^2+y^2}$$

$$= \lim_{(y,y)\to(0,0)} \frac{4y^2}{2y^2} = \underline{2}$$

Since the limits are different when approaching from different direction, the overall limit does not exist.

Excellent.

5. Write an equation for the plane passing through the points (5,0,1), (-2,2,3), and the origin.

$$\underset{P}{(5,0,1)} \quad \underset{Q}{(-2,2,3)} \quad \underset{R}{(0,0,0)}$$

$$\overrightarrow{PR} = (0-5)\vec{i} + (0-0)\vec{j} + (0-1)\vec{k}$$

$$= -5\vec{i} + 0\vec{j} - \vec{k}$$

$$\overrightarrow{QR} = (0+2)\vec{i} + (0-2)\vec{j} + (0-3)\vec{k}$$

$$= 2\vec{i} - 2\vec{j} - 3\vec{k}$$

$$\overrightarrow{PR} \times \overrightarrow{PQ} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -5 & 0 & -1 \\ 2 & -2 & -3 \end{vmatrix}$$

$$\begin{array}{r} 0\vec{i} - 2\vec{j} + 10\vec{k} \\ - \quad 2\vec{i} + 15\vec{j} \quad 0\vec{k} \\ \hline -2\vec{i} - 17\vec{j} + 10\vec{k} \end{array}$$

$$0 = -2(x-0) - 17(y-0) + 10(z-0)$$

$$0 = -2x - 17y + 10z$$

Wonderful!

$$z = \frac{2}{10}x + \frac{17}{10}y \longrightarrow z = \frac{1}{5}x + \frac{17}{10}y$$

W

✓ $0 = -2(x+2) - 17(y-2) + 10(z-3)$

$0 = -2x - 4 - 17y + 34 + 10z - 30$

$0 = -2x - 17y + 10z$

$-10z = -2x - 17y$

$\frac{10z}{10} = \frac{2x + 17y}{10}$

$z = \frac{1}{5}x + \frac{17}{10}y$ ✓

6. Show that for any vectors $\vec{a}$ and $\vec{b}$, the vector $\vec{a}\times\vec{b}$ is perpendicular to $\vec{b}$.

$\vec{a} = \langle a_1, a_2, a_3 \rangle$ $\quad \vec{b} = \langle b_1, b_2, b_3 \rangle$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \begin{matrix} - a_2b_3\vec{i} + a_3b_1\vec{j} + a_1b_2\vec{k} \\ \underline{a_3b_2\vec{i} + a_1b_3\vec{j} + a_2b_1\vec{k}} \end{matrix}$$

$$\langle a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1 \rangle$$

To see if $\vec{a}\times\vec{b}$ is $\perp$ to $\vec{b}$ we dot them

$(\vec{a}\times\vec{b})\cdot\vec{b}$

$\langle a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1 \rangle \langle b_1, b_2, b_3 \rangle$

$= (a_2b_3 - a_3b_2)b_1 + (a_3b_1 - a_1b_3)b_2 + (a_1b_2 - a_2b_1)b_3$

$= \cancel{b_1a_2b_3} - \cancel{b_1a_3b_2} + \cancel{b_2a_3b_1} - \cancel{b_2a_1b_3} + \cancel{a_1b_2b_3} - \cancel{a_2b_1b_3}$

$= 0$

Since their dot product is 0, they are perpendicular ∎

Great Job!

7. Bunny is a calculus student at Enormous State University, and she's having some trouble. Bunny says "Ohmygod, this is the most totally confusing experience in my life. The professor told us there were these things we definitely had to know for the test, like in my notes I have that she said that the level curvy things are ninety degrees from the direction of greatest increase. And she said we have to know why that's true, but I totally don't have a clue. I looked in the book and it makes no sense at all. She never said anything about it in class, just during the review. So how am I supposed to know why it's true? This is so unfair!"

Explain clearly to Bunny how she could deduce such a conclusion from other things which she should indeed know.

I'm not sure either what's being said here.

Oh wait, okay this is talking about gradients and the gradient is in the direction of the greatest increase.

$f_{\vec{u}} = \|\nabla f\| \cdot \|\vec{u}\| \cos\theta$

So maybe the 90° the teacher is talking about means 90° in this formula

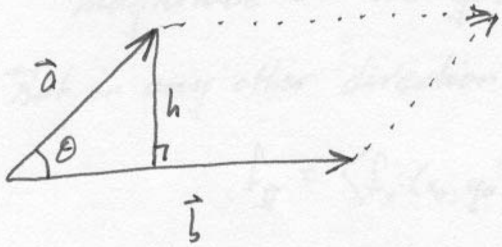
and the $\cos 90° = 0$

That makes the whole thing 0 and that would mean 0 increas, giving level curve.

Yes!

8. It was stated in class that $\|\vec{a}\times\vec{b}\|$ gives the area of the parallelogram with sides $\vec{a}$ and $\vec{b}$.

Explain why this is true. [Feel free to use the fact that $\|\vec{a}\times\vec{b}\|=\|\vec{a}\|\|\vec{b}\|\sin\theta$, even though we didn't actually prove it in class either.]



Well, what we really need is the length marked h in the diagram at left, since it times the length of $\vec{b}$ will give the area of the figure.

But since $\sin\theta = \frac{h}{\|\vec{a}\|}$, or $h = \|\vec{a}\|\sin\theta$,

We have that Area $= \|\vec{b}\|\cdot h$ or

$$\text{Area} = \|\vec{b}\|\cdot\|\vec{a}\|\sin\theta,$$

as desired. $\square$

9. Suppose that you're standing at a point (x_0, y_0) on the graph of $f(x, y)$. In which direction(s) is the slope of the surface equal to half of the greatest slope at that point?

Well, the greatest slope at that point is given by the magnitude of the gradient, which is $\|\langle f_x(x_0,y_0), f_y(x_0,y_0)\rangle\|$.

But in any other direction the slope is given by

$$f_{\vec{u}} = \langle f_x(x_0,y_0), f_y(x_0,y_0)\rangle \cdot \vec{u}$$

$$= \|\langle f_x(x_0,y_0), f_y(x_0,y_0)\rangle\|\cdot\|\vec{u}\|\cdot\cos\theta$$

where θ is the angle formed between $\vec{u}$ and the gradient.

So for this directional derivative to be half as big as the magnitude of the gradient,

$$\|\langle f_x(x_0,y_0), f_y(x_0,y_0)\rangle\|\cos\theta = \tfrac{1}{2}\|\langle f_x(x_0,y_0), f_y(x_0,y_0)\rangle\|$$

or just

$$\cos\theta = \tfrac{1}{2}.$$

But this is true exactly when $\theta = \pm\frac{\pi}{3}$, or when you're facing $60°$ off (either way) from the direction of the gradient. $\square$

10. Suppose f is a differentiable function of one variable. Show that all tangent planes to the surface $z = x\,f(y/x)$ intersect in a common point (Stewart 5th, p. 978). [Hint$_1$: Warm up with a simple function, like $f(x) = x^2$. Hint$_2$: If you were lucky, where would the common point be?.]

Well, I want a tangent plane to $z = x \cdot f\left(\frac{y}{x}\right)$, so I'll need partials:

$$z_x = 1 \cdot f\left(\tfrac{y}{x}\right) + x \cdot f'\left(\tfrac{y}{x}\right) \cdot -yx^{-2} \qquad z_y = x \cdot f'\left(\tfrac{y}{x}\right) \cdot \tfrac{1}{x}$$

$$z_x = f\left(\tfrac{y}{x}\right) + \tfrac{-y}{x} f'\left(\tfrac{y}{x}\right) \qquad z_y = f'\left(\tfrac{y}{x}\right)$$

So the plane tangent to this surface at (x_0, y_0, z_0) is:

$$z - x_0 f\left(\tfrac{y_0}{x_0}\right) = \left[f\left(\tfrac{y_0}{x_0}\right) + \tfrac{-y_0}{x_0} f'\left(\tfrac{y_0}{x_0}\right)\right](x - x_0) + f'\left(\tfrac{y_0}{x_0}\right)(y - y_0)$$

Yikes. But what I need to know is if all such planes share a common point. My only hope is that it's a nice point, like maybe the origin. So let's see if $(0,0,0)$ is on this plane by substituting it in:

$$(0) - x_0 f\left(\tfrac{y_0}{x_0}\right) \overset{?}{=} \left[f\left(\tfrac{y_0}{x_0}\right) - \tfrac{y_0}{x_0} f'\left(\tfrac{y_0}{x_0}\right)\right]((0) - x_0) + f'\left(\tfrac{y_0}{x_0}\right)((0) - y_0)$$

$$-x_0 f\left(\tfrac{y_0}{x_0}\right) \overset{?}{=} -x_0 f\left(\tfrac{y_0}{x_0}\right) + \cancel{y_0 f'\left(\tfrac{y_0}{x_0}\right)} - \cancel{y_0 f'\left(\tfrac{y_0}{x_0}\right)}$$

$$-x_0 f\left(\tfrac{y_0}{x_0}\right) \overset{?}{=} -x_0 f\left(\tfrac{y_0}{x_0}\right) \checkmark$$

So since that equation is true for any (x_0, y_0) in the domain of f, we conclude that all tangent planes have a common point of intersection – namely the origin! Yay!