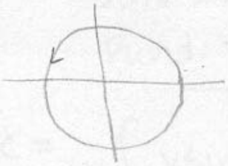
Exam 3 Calc 3 12/2/2005

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Give a parametrization and bounds for a counterclockwise circle centered at the origin in the *xy*-plane with radius 3.



$x(t) = 3\cos t$

$y(t) = 3\sin t$

$0 \le t \le 2\pi$

Yep.

2. Find the divergence of the vector field $\mathbf{F}(x,y,z) = \langle -z, y^2, x\rangle$.

$$\text{Div } \vec{F} = \nabla \cdot F = \frac{\partial(-z)}{\partial x} + \frac{\partial(y^2)}{\partial y} + \frac{\partial(x)}{\partial z}$$

$$= \boxed{2y}$$

Good

3. Evaluate $\int_C \langle \overset{P}{y\sin x}, \overset{Q}{-\cos x}\rangle \cdot d\mathbf{r}$ where C is a line segment beginning at (0,1) and ending at $(2\pi,2)$.

sinx sinx

WOO HOO! There is a potential since the partials equal each other! So we can use the Fundemental Theorum for line integrals

$f(b) - f(a)$

$\boxed{\text{Potential} = -y\cos x}$

10

$[-y\cos x]$ $f(2\pi,2) - f(0,1)$ Excellent!

$-2\cos(2\pi) + + \cos(0)$

$-2+1 = \boxed{-1}$

4. Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F}(x,y,z) = 5x\mathbf{i} - 2y\mathbf{j} + 7z\mathbf{k}$ and S is the surface of the outward-oriented cube bounded by the coordinate planes and the planes $x = 1$, $y = 1$, and $z = 1$.

closed Surface use divergence theorem

unit cube

$$\iint \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F}\, dV$$

$$\int_0^1\int_0^1\int_0^1 \operatorname{div} \vec{F}\, dx\,dy\,dz \qquad \operatorname{div} \vec{F} = 5 + -2 + 7 = 10$$

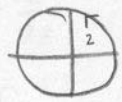
$$\int_0^1\int_0^1\int_0^1 10\, dx\,dy\,dz = 10 \int_0^1\int_0^1\int_0^1 1\, dx\,dy\,dz = 10$$

volume of unit cube since triple with integrand of 1 = 1

Well done

5. Let $\mathbf{F} = (z - 2y)\mathbf{i} + (3x - 4y)\mathbf{j} + (z + 3y)\mathbf{k}$, and let C be the circle $x^2 + y^2 = 4$, $z = 1$, oriented counterclockwise when viewed from above. **Use Stokes' Theorem** to find $\int_C \mathbf{F} \cdot d\mathbf{r}$.

$F = \langle z - 2y,\ 3x - 4y,\ z + 3y \rangle$



$\int_C \vec{F} \cdot d\vec{r} =$

$$\begin{array}{ccc} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z - 2y & 3x - 4y & z + 3y \end{array}$$

$3\vec{i} + 3k + 1\vec{j} + 2k - 0i + 0j$

$\iint_S \langle 3, 1, 5 \rangle\, d\vec{S} \qquad \text{curl} = \langle 3, 1, 5 \rangle$

$$\int_0^{2\pi}\int_0^2 \overset{\text{curl}}{\langle 3, 1, 5 \rangle}\, d\vec{S}$$

only has a normal vector in the K spot so you multiply the K(5) with the area of the circle $\pi r^2 = 4\pi \cdot 5 = 20\pi$

Yes!

6. Show that curl(grad $f(x,y,z)$) = **0** for any function $f(x,y,z)$ with continuous second-order partials. Make it clear why it matters that the second-order partials be continuous.

$$\nabla f(x,y,z) = \langle f_x, f_y, f_z \rangle$$

$$\text{curl}(\nabla f) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix}$$

$$= f_{zy}\vec{i} + f_{yx}\vec{k} + f_{xz}\vec{j} - f_{xy}\vec{k} - f_{yz}\vec{i} - f_{zx}\vec{j}$$

$$= \underline{\langle f_{zy} - f_{yz}, f_{xz} - f_{zx}, f_{yx} - f_{xy} \rangle}$$

Since the second order partials is continuous, they are equal

$$f_{zy} = f_{yz}$$
$$f_{xz} = f_{zx}$$
$$f_{yx} = f_{xy}$$

Nice!

therefore they cancel each other

$$\underline{\text{curl}(\nabla f) = \langle 0, 0, 0 \rangle = \vec{0}}$$

7. Let $\mathbf{F}(x,y,z) = \langle -y, x\rangle$ and let C be a counterclockwise circle with radius a and center (h,k). Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ in terms of the values of a, h, and k.

Closed path → use Green's Theorem

$$\int_C \vec{F}\cdot d\vec{r} = \iint_D \left(\frac{dQ}{dx} - \frac{dP}{dy}\right) dA = \iint_D (1 - -1)\, dA = \iint_D 2\, dA$$

2 times area of the circle (πa^2) so $2\pi a^2$ is the answer

Excellent

9. Let $\mathbf{F}(x,y,z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, and let S be the portion of the outward-oriented surface $x^2 + y^2 = 4$ between the planes $z = a$ and $z = a + h$. Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$. not closed, do long way

$\vec{r}(u,v) = \langle 2\cos u, 2\sin u, v \rangle$

$x(u,v) = 2\cos u$

$y(u,v) = 2\sin u$

$z(u,v) = v$

$0 \le u \le 2\pi$

$a \le v \le a+h$

$\vec{r}_u = \langle -2\sin u, 2\cos u, 0 \rangle$

$\vec{r}_v = \langle 0, 0, 1 \rangle$

$\vec{r}_u \times \vec{r}_v = \langle 2\cos u, 2\sin u, 0 \rangle$

$\vec{F}(\vec{r}(u,v)) = \langle 2\cos u, 2\sin u, v \rangle$

$$\int_0^{2\pi}\int_a^{a+h} \langle 2\cos u, 2\sin u, v \rangle \cdot \langle 2\cos u, 2\sin u, 0 \rangle \, dv\, du$$

$$\int_0^{2\pi}\int_a^{a+h} (4\cos^2 u + 4\sin^2 u)\, dv\, du$$

$$\int_0^{2\pi}\int_a^{a+h} (4)\, dv\, du$$

$$\int_0^{2\pi} \left[4v\right]_a^{a+h} du$$

Wonderful

$$\int_0^{2\pi} (\cancel{4a} + 4h - \cancel{4a})\, du$$

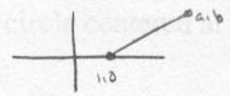
$$\left[4hu\right]_0^{2\pi}$$

$8\pi h$

10. Let $\mathbf{F}(x,y) = \langle -y, x \rangle$. Let C be a line segment starting at (1,0) and ending at (a,b) for some constants a and b. Which values can a and b have in order for $\int_C \mathbf{F} \cdot d\mathbf{r}$ to work out to 3?

No potential

Not closed



I. $x(t) = 1 + (a-1)t$

$y(t) = bt$ $\qquad 0 \le t \le 1$

$\vec{r} = \langle 1+(a-1)t,\ bt \rangle$

II. $F(\vec{r}(t)) = \langle -bt,\ 1+(a-1)t \rangle$ $\qquad 1+at-t,\ bt$

III. $\vec{r}\,'(t) = \langle a-1,\ b \rangle$

IV. $\int_0^1 \langle a-1, b \rangle \cdot \langle -bt,\ 1+at-t \rangle \, dt$

$= \left(-bt(a-1) + b + bat - bt\right) dt$

$= \left(-bta + bt + b + bat - bt\right) dt$

$= \left(-bat + bt + b + bat - bt\right) dt$

$= b$

$\int_0^1 b \, dt$ $\qquad$ b has to equal 3 $\qquad$ Exactly.

a doesn't matter