Exam 1 Calc 2 9/15/2006

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Evaluate $\int (\sin x - x^3 + x^{-1} + e^x)\,dx$.

$$= \boxed{-\cos x - \frac{x^4}{4} + \ln|x| + e^x + C} = f(x)$$

check: $f'(x) = \sin x - x^3 + \frac{1}{x} + e^x$

$= \sin x - x^3 + x^{-1} + e^x$ ✓

Great

2. **Set up an integral** for the average value of the function $g(t) = x^2$ on the interval [0,2].

$$A = \frac{1}{b-a}\int_a^b f(x)\,dx$$

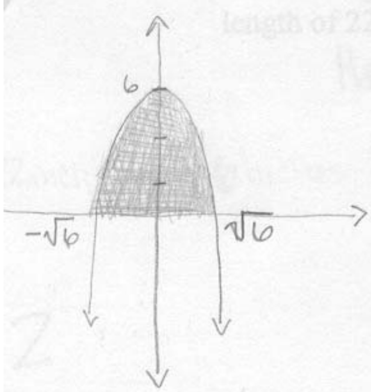
$$A = \frac{1}{2-0}\int_0^2 x^2\,dx$$

$$A = \frac{1}{2}\int_0^2 x^2\,dx$$

Great

3. Find the area of the region between $y = 6 - x^2$ and the x-axis.

& $y = 0$



$6 - x^2 = 0$

$x = \pm\sqrt{6}$

Excellent!

$$\text{Area} = \int_{-\sqrt{6}}^{\sqrt{6}} (6 - x^2)\,dx$$

$$= \left[6x - \frac{x^3}{3}\right]_{-\sqrt{6}}^{\sqrt{6}}$$

$$= \left[6(\sqrt{6}) - \frac{(\sqrt{6})^3}{3}\right] - \left[6(-\sqrt{6}) - \frac{(-\sqrt{6})^3}{3}\right]$$

$$= 4\sqrt{6} - (-4\sqrt{6}) = \boxed{8\sqrt{6}}$$

4. Evaluate $\int \frac{1}{x(\ln x)^3}\,dx$.

$\ln x = u$

$\frac{1}{x} = \frac{du}{dx}$

$\frac{1}{x}\,dx = du$

$= \int \frac{1}{u^3}\,du$

$= \int u^{-3}\,du$

$= -\frac{1}{2}u^{-2} + C$

$= \boxed{-\frac{1}{2(\ln x)^2} + C}$

Excellent

NL $= 4/3$ ft.

$w = fd$

$f = kx$ → force

$\frac{20-16}{12} = x$

5. A spring has a natural length of 16 inches, and 60 pounds hold it stretched to a length of 20 inches. How much work is done in stretching the spring from a length of 20 inches to a length of 22 inches?

$x = 1/3$ ft → bottom limit $= 1/3$ ft.

↳ top limit $= 1/2$

$60\text{ lbs} = k \cdot \frac{1}{3}\text{ ft}$

$180 \frac{\text{lbs}}{\text{ft}} = k$

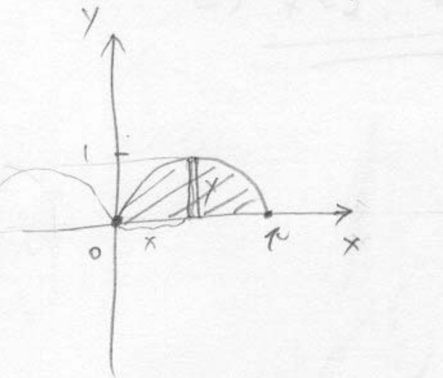
$W = \int_{1/3}^{1/2} 180\,x\,dx = 180 \int_{1/3}^{1/2} x\,dx$

$= 180\left[\frac{x^2}{2}\right]_{1/3}^{1/2} = 90\left(\frac{1}{4} - \frac{1}{9}\right) = \frac{25}{2} = \boxed{12.5 \text{ ft lbs}}$

Great

Answer: 12.5 ft lbs

6. Suppose that the region bounded by $y = \sin x$ and the x-axis between $x = 0$ and $x = \pi$ is rotated around the y-axis. **Set up** an integral for the volume of the resulting solid.



rotated around y-axis

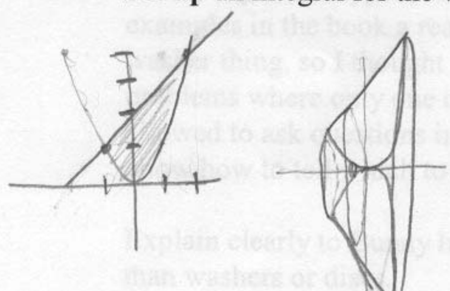
radius = x

height = $y = \sin x$

$$V = \int_0^{\pi} \underbrace{2\pi(x)}_{\text{radius}} \cdot \underbrace{\sin x}_{\text{height}} \, dx$$

Excellent

7. Suppose that the region bounded between $y = x^2$ and $y = x + 2$ is rotated around the x-axis. **Set up** an integral for the volume of the resulting solid.



outer radius $= x+2$

inner radius $= x^2$

Area of Washer $= \pi(x+2)^2 - \pi(x^2)^2$

$\pi((x+2)^2 - (x^2)^2)$

Volume of washer $= \pi((x+2)^2 - (x^2)^2)\Delta x$

$[-1, 2]$

Total Volume =

$$\pi \int_{-1}^{2} ((x+2)^2 - x^4)\,dx$$

Excellent

8. Bunny is a calculus student at Enormous State University, and she's having some trouble. Bunny says "Ohmygod, this is the most totally confusing experience in my life. I studied the examples in the book a really long time and figured out how to do the shells thing and the washer thing, so I thought I was pretty good. But then our professor said something about problems where only one of them works, and I have no idea how you tell that, and we're not allowed to ask questions in lecture because there's like a thousand people. The only way I know how to tell which to use is by which section the homework problem is from!"

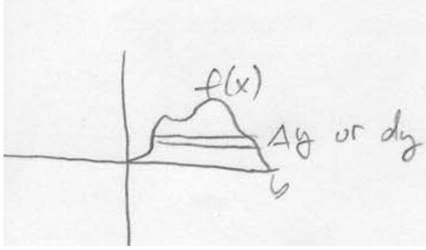
Explain clearly to Bunny how you can tell when a problem needs to by done by shells rather than washers or discs.

Well Bunny, a clear example of when to use Shells instead of washers or discs is when it is difficult to solve functions that are given to you in terms of x, but you need them in terms of y.

For example, lets say you are rotating some function around the y-axis (see fig. 1). Lets assume the function is given to us in terms of x, but is impossible to solve explicitly in terms of y. By using washers you must have the equation in terms of y b/c each one of the infinite number of washers has a width of Δy or dy. To use washers is impossible and impraticitable.

Now shells would be easy here.

The radius is x, so the circumference is $2\pi x$ and the height is $f(x)$

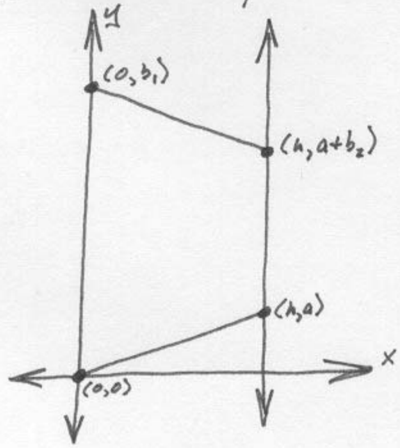


$$\therefore \int_0^b 2\pi x\, f(x)\, dx$$

Wonderful

9. Set up and evaluate an integral to show that the area of a trapezoid with height h and parallel sides with lengths b_1 and b_2 has area $\dfrac{h(b_1+b_2)}{2}$.

Let's line it up like this:



So the top line has

$$m=\frac{(a+b_2)-(b_1)}{(h)-(0)}=\frac{a+b_2-b_1}{h}$$

and equation

$$y-(b_1)=\frac{a+b_2-b_1}{h}(x-(0))$$

$$y=\frac{a+b_2-b_1}{h}x+b_1$$

And the bottom line has

$$m=\frac{(a)-(0)}{(h)-(0)}=\frac{a}{h}$$

and equation

$$y-(0)=\frac{a}{h}(x-(0))$$

$$y=\frac{a}{h}x$$

So the area is given by:

$$\int_0^h\left[\left(\frac{a+b_2-b_1}{h}x+b_1\right)-\left(\frac{a}{h}x\right)\right]dx=\left[\frac{a+b_2-b_1}{h}\cdot\frac{x^2}{2}+b_1x-\frac{a}{h}\cdot\frac{x^2}{2}\right]_0^h$$

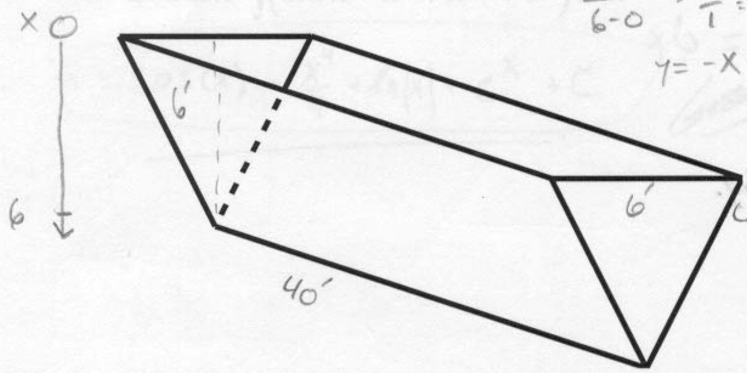
$$=\frac{a+b_2-b_1}{h}\cdot\frac{h^2}{2}+b_1h-\frac{a}{h}\cdot\frac{h^2}{2}-0$$

$$=\frac{ah}{2}+\frac{b_2h}{2}-\frac{b_1h}{2}+\frac{2b_1h}{2}-\frac{ah}{2}$$

$$=\frac{b_2h}{2}+\frac{b_1h}{2}$$

$$=\frac{h(b_1+b_2)}{2}$$

10. Suppose that a ditch shaped like a triangular prism (shown below) needs to be drained in order to free a baby giraffe whose leg is stuck in a culvert at one end of the ditch. The ditch is 40 feet long, 6 feet wide, and 6 feet deep, full of water weighing 62.5 pounds per ft^3. How much work is required to pump all the water up to the level of the top of the ditch?



$\frac{0-6}{6-0} = \frac{-1}{1} = -1$

$y = -x + 6$

length of a side - $(-x+6)$ and 40

area of a slice - $(-x+6)(40)$ ft^2

Volume of a slice - $(-x+6)(40)\Delta x$ ft^3

Force of a slice - $(-x+6)(40)\Delta x$ ft^3 (62.5) lbs/ft^3

Work of a slice - $(-x+6)(40)\Delta x\,(62.5)$ lbs (x) ft

Total work

$$\int_0^6 62.5x(-x+6)(40)\,dx$$

$$2500\int_0^6 (x)(-x+6)$$

$$2500\int_0^6 -x^2+6x$$

$$(2500)\left(-\frac{x^3}{3}+3x^2\right)\Big|_0^6$$

$$2500\left(\frac{-(6)^3}{3}+3(6)^2\right)$$

$$2500(-72+108)$$

$$2500(36) = \boxed{90{,}000\text{ ft}\cdot\text{lbs}}$$

Well done