Exam 4 Calc 2 12/1/2006

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Give the 5th degree MacLaurin polynomial for sin x.

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

so $$P_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

or $$= x - \frac{x^3}{6} + \frac{x^5}{120}$$

2. Find the sum $\frac{1}{3} - \frac{1}{9} + \frac{1}{27} - \frac{1}{81} + \ldots$

It's a geometric series so $S = \frac{a}{1-r}$

$a = \frac{1}{3}$ $r = -\frac{1}{3}$

where $|-\frac{1}{3}| < 1$

$$S = \frac{\frac{1}{3}}{1 + \frac{1}{3}}$$

$$S = \frac{\frac{1}{3}}{\frac{4}{3}} \quad \frac{1}{3} \cdot \frac{3}{4} = \frac{3}{12}$$

Great

$$S = \frac{1}{4}$$

By the geometric series test.

3. Determine whether the series $\sum_{n=0}^{\infty} \frac{n}{n+1}$ converges or diverges.

Use Test for divergence!

$$\lim_{n\to\infty} \frac{n}{n+1} \overset{L'H}{=} \frac{1}{1} \neq 0$$

$\boxed{\text{Diverges}}$

Excellent

4. Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{5n+3}$ converges or diverges.

compare to $b_n = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{1}{5n+3} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{n}{5n+3} \overset{L'H}{=} \lim_{n\to\infty} \frac{1}{5} = \frac{1}{5}$$

Since the series converges to a finite, pos. #

$\sum_{n=1}^{\infty} \frac{1}{5n+3}$ diverges by the Limit comparison test

b/c $\sum_{n=0}^{\infty} \frac{1}{n}$ diverges by the p-series test where for $p < 1$

the series diverges

(harmonic series)

Excellent

5. Determine whether the series $\sum_{n=0}^{\infty} \frac{1}{3+n!}$ converges or diverges. *Comparison!*

I know $\sum \frac{1}{n!}$ converges, since it's the series for e^x (at $x=1$) and we know that has infinite radius of convergence. Also

$3 > 0$ (2nd grade)

$3+n! > n!$ (adding)

$\frac{1}{n!} > \frac{1}{3+n!}$ (dividing by positives)

So $\frac{1}{3+n!}$ is less than $\frac{1}{n!}$, and $\sum \frac{1}{n!}$ converges, thus $\sum \frac{1}{3+n!}$ must converge also.

6. Suppose that you begin with the closed interval [0,1] and divide it into five equal parts, then discard the second and fourth open intervals, i.e. (1/5, 2/5) and (3/5, 4/5). Next take each of the remaining intervals and divide each into five equal parts, discarding the second and fourth open intervals as before. If this process is repeated indefinitely, what is the total length of the intervals removed?

The first step removes 2 intervals each of length $\frac{1}{5}$, so a total of $\frac{2}{5}$

The second step removes $2 \cdot 3$ intervals each of length $\frac{1}{25}$, so a total of $\frac{6}{25}$

The third step removes $2 \cdot 3 \cdot 3$ intervals each of length $\frac{1}{125}$, so a total of $\frac{18}{125}$

Each subsequent step involves 3 times as many intervals, each $\frac{1}{5}$ as long as in the previous step, so it's a geometric sequence with first term $2/5$ and the ratio between successive terms is $3/5$.

Thus $S = \frac{2/5}{1-(3/5)} = \frac{2/5}{2/5} = 1$

7. For which values of p will the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ converge, and why?

Use the Alternating Series Test!

- Sign alternates because of $(-1)^n$
- Decreasing because $(n^{-p})' = -p \cdot n^{-p-1} < 0$
- $\lim_{n\to\infty} \frac{1}{n^p} = 0$ as long as $p > 0$, but diverges for $p < 0$ or $= 1$ for $p = 0$ (so that the series then diverges by the Divergence Test)

So $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ converges exactly when $p > 0$

8. Biff is a calculus student at Enormous State University, and he's still having some trouble. Biff says "Dude, this series stuff is killing me. Especially these questions where you're supposed to say what it adds up to, I mean what's up with that? When it's just diverges or converges, it's fifty-fifty, and I like those odds, ya know? But there was this question on our test where we were supposed to say what one over *e* to the *n* adds up to, and I did it with the integral test and got 1 over *e*, so I said 1 over *e*, and the grader wrote a bunch of bad stuff about how I couldn't use that, but I dunno why."

Explain clearly to Biff what was wrong with using the results of the integral test as he did.

Biff, you're mixing things together. The integral test just tells you whether a series converges or not, rather than what it converges to. The fact that you got a finite number like $\frac{1}{e}$ means $\sum \frac{1}{e^n}$ does converge, but doesn't tell you to what.

If you need to know what the sum actually is, your best option is to take a fresh look and notice that $\sum \left(\frac{1}{e}\right)^n$ is a geometric series. That means you can use the formula for the sum of a geometric series to find the total.

And by the way, it's also a good idea to write out a few terms in a sequence right away just to get a feel for it. Since

$$\sum_{n=1}^{\infty} \frac{1}{e^n} = \frac{1}{e} + \frac{1}{e^2} + \frac{1}{e^3} + \dots$$

it would be pretty amazing for it to add up to $\frac{1}{e}$ since it started with that and kept getting more!

9. Use a power series with at least 3 non-zero terms to approximate $\int_0^{0.1} \frac{1}{1-x^5}\,dx$.

I know:

$$\frac{1}{1-x} \approx 1 + x + x^2$$

So

$$\frac{1}{1-x^5} \approx 1 + (x^5) + (x^5)^2 = 1 + x^5 + x^{10}$$

Then

$$\int_0^{0.1} \frac{1}{1-x^5}\,dx \approx \int_0^{0.1} \left(1 + x^5 + x^{10}\right) dx = \left[x + \frac{x^6}{6} + \frac{x^{11}}{11} \right]_0^{0.1}$$

$$= (0.1) + \frac{(0.1)^6}{6} + \frac{(0.1)^{11}}{11} - 0$$

$$\approx 0.100000166668$$

10. The radius of convergence of the power series $\displaystyle\sum_{n=2}^{\infty}\frac{x^n}{\ln n}$ is 1. Are the endpoints included in the interval of convergence?

When $x=1$: $\displaystyle\sum_{n=2}^{\infty}\frac{(1)^n}{\ln n}=\sum_{n=2}^{\infty}\frac{1}{\ln n}$

And since $n > \ln n$,

$$\frac{1}{\ln n} > \frac{1}{n}$$

so because $\sum \frac{1}{n}$ diverges (it's the Harmonic Series), $\sum \frac{1}{\ln n}$ also diverges by the Comparison Test

When $x=-1$: $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^n}{\ln n}$

- Alternating because of $(-1)^n$
- $\displaystyle\lim_{n\to\infty}\frac{1}{\ln n}=0$
- $\displaystyle\left(\frac{1}{\ln n}\right)'=\frac{0\cdot\ln n-1\cdot\frac{1}{n}}{(\ln n)^2}$

$$=-\frac{1}{n(\ln n)^2}<0$$ since n is positive and squares are non-negative, so it's decreasing.

So by the Alternating Series Test, this series converges for $x=-1$,

$\therefore$ 1 is not in the interval of convergence, but

-1 is in the interval of convergence.