Exam 3 Calc 3 12/1/2006

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Give a parameterization, along with bounds, for the line segment from the origin to the point (2, –3, 5). $(0,0,0)$

$x(t) = (2-0)t = 2t$
$y(t) = (-3-0)t = -3t$ for $0 \le t \le 1$
$z(t) = (5-0)t = 5t$

Good

2. Compute div $\langle 3, 4x^2y, 2y^5 \cos x \rangle$.

divergence $\vec{F}: \nabla \cdot F$

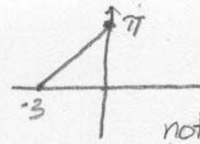
$\nabla = \langle d/dx, d/dy, d/dz \rangle$

div: $\langle d/dx, d/dy, d/dz \rangle \cdot \langle 3, 4x^2y, 2y^5 \cos x \rangle$

$= \frac{d3}{dx} + \frac{d4x^2y}{dy} + \frac{d\, 2y^5 \cos x}{dz}$

$0 + 4x^2 + 0 = \boxed{4x^2}$

Great



not closed

line integral

3. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y) = 2x \cos y\, \mathbf{i} - x^2 \sin y\, \mathbf{j}$ and C is the line segment from (-3,0) to $(0,\pi)$.

potential function $\rightarrow x^2 \cos y$

Use Fun. Thm for Line Integrals.

$$\left[x^2 \cos y\right]_{(-3,0)}^{(0,\pi)} = 0^2(\cos\pi) - (-3)^2\cos(0)$$

$$= 0 - 9(1)$$

$$= -9$$

Well done

4. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y) = \langle 3y, 5x \rangle$ and C is the circle centered at the origin with radius 7, traversed clockwise. $\rightsquigarrow$ neg. orientation

P Q

closed path so use Green's Theorem

$$\oint_C P\,dx + Q\,dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$$

$$\frac{\partial Q}{\partial x} = 5 \quad \frac{\partial P}{\partial y} = 3$$

$\iint_D 5 - 3\,dA = \iint_D 2\,dA$ then find area of circle

πr^2 when $r = 7$ $\quad \pi 7^2 = 49\pi$

and times by two

$2 \cdot 49\pi = 98\pi$ but since it is not positively oriented (going clockwise) we get $\boxed{-98\pi}$

Excellent!

5. Compute $\int_C \mathbf{G} \cdot d\mathbf{r}$ where $\mathbf{G}(x, y) = \langle 6xy, 2y^2 \rangle$ and C is the left half of a unit circle centered at the origin and traversed counterclockwise.

not closed

$f_y(6xy) \quad f_x(2y^2)$

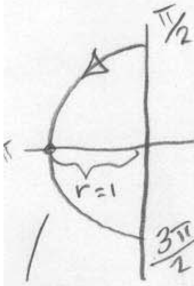
$6x \neq 0$

no pot. fun

no green's

I parametrize

long way (yuck, but someone's gotta do it)



Unit circle

$x = \cos t$

$y = \sin t$

$\frac{\pi}{2} \le t \le \frac{3\pi}{2}$

$\vec{r} = \langle \cos t, \sin t \rangle$

$\vec{r}\,' = \langle -\sin t, \cos t \rangle$

$\vec{F}(\vec{r}) = \langle 6\cos t \sin t, 2\sin^2 t \rangle$

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \langle 6\cos t \sin t, 2\sin^2 t \rangle \cdot \langle -\sin t, \cos t \rangle$$

$= -6\cos t \sin^2 t + 2\sin^2 t \cos t$

$\cos t(-6\sin^2 t + 2\sin^2 t)$

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} -4\sin^2 t \cos t \rightarrow \text{u sub}$$

$u = \sin t$

$du = \cos t\,dt$

$\frac{du}{\cos t}$

$-4u^2 \frac{du}{\cancel{\cos t}} \cancel{\cos t}$

10

well done

$\int -4u^2\,du$

$-\frac{4}{3}\sin^3 t \Big|_{\pi/2}^{3\pi/2} \rightarrow \left(\frac{4}{3}\right) - \left(\frac{-4}{3}\right) = \frac{8}{3}$

$\frac{4}{3} + \frac{4}{3}$

6. Show that curl ($\nabla f(x, y, z)$) = **0** provided that the mixed second-order partials of f are continuous. Make it clear how the requirement about the partials is involved.

$$\nabla f(x,y,z) = \left\langle \frac{df}{dx}, \frac{df}{dy}, \frac{df}{dz} \right\rangle$$

$$\text{curl}(\nabla f) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix}$$

$$= (f_{zy} - f_{yz})\vec{i} - (f_{zx} - f_{xz})\vec{j} + (f_{yx} - f_{xy})\vec{k}$$

$$= \langle f_{zy} - f_{yz},\ f_{xz} - f_{zx},\ f_{yx} - f_{xy} \rangle$$

Since the second-order partials are continuous, by Clairout's Theorem items like $f_{zy} = f_{yz}$. This means that in the above vector all the partials will cancel leaving you with the zero vector.

$$\boxed{\text{curl}(\nabla f) = \vec{0}}$$

Excellent

7. Bunny is a Calc 3 student at Enormous State University, and she's having some trouble. Bunny says "Ohmygod! I've never been so confused in my life! I though math was supposed to be about *numbers*, right? But now there's all this stuff with closed and stuff like that, and I missed the lecture about it because I had to get my hair cut. I thought it was things like doors and windows that could be closed, not *math*! It's totally unfair. I can't figure out how I'm supposed to know when something is closed or when it isn't, and I guess it's supposed to matter, right? So how do you tell if something's closed?"

Explain clearly to Bunny what sorts of things are or aren't closed, and how you can tell.

Bunny,

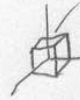
Firstly something is closed in two dimensions when the path of the line ends in the same place as it started as long as the line does not cross itself.

Good.

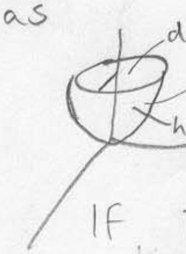
In three dimensions, a closed object is something that a metaphorical kitten could not get into by finding a gap in the surface. Basic examples are.

A sphere

a cube that includes all of its sides



But it also extends to objects with lids on them such as



dis hyperboloid

the surface of a hyperboloid capped off by a disc including the disc

discs

The surface of a cone capped off by two discs. but including the discs

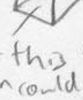
If the discs are not included in examples like this such as

$z=4$

kitten could get in here

a hyperboloid cut of by the plane $z=4$

the surface of 5 sides of a cube



not this side (kitten could get in this side)

A cone cut off by the planes $z=1$ & $z=4$

the surface is not closed because the kitten could get into the object through the empty space

8. Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F}(x, y, z) = x\,\mathbf{i} - z\,\mathbf{k}$ and S consists of the paraboloid $x = y^2 + z^2$ for $x \le 4$ and the disk $y^2 + z^2 \le 4$ for $x = 4$.

Closed surface → use Divergence theorem:

$\text{div } \vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(-z)$

$\text{div } \vec{F} = 1 + 0 - 1$

$\text{div } \vec{F} = 0$

$\iiint \text{div } \vec{F} \, dV$

$\iiint 0 \, dV$ & $\text{div } \vec{F} = 0$ so the integral $= 0$

$= 0$

Good.

9. Evaluate $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F}(x, y, z) = xyz\,\mathbf{i} + xy\,\mathbf{j} + x^2yz\,\mathbf{k}$, and S consists of the top and four sides (but not the bottom) of the cube with vertices $(\pm 1, \pm 1, \pm 1)$, oriented outward.

It's a curl field, so use Stokes' Theorem.

But instead of actually doing that line integral (around all four edges of the bottom face) let's use Stokes' again, backwards, to trade that for the flux through the bottom face with downward orientation.

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & xy & x^2yz \end{vmatrix} = \left(x^2z\,\vec{i} + xy\,\vec{j} + y\,\vec{k}\right) - \left(0\,\vec{i} + 2xyz\,\vec{j} + xz\,\vec{k}\right)$$

$$= \langle x^2z,\ xy - 2xyz,\ y - xz \rangle$$

I. $x(u,v) = u$

$y(u,v) = v$

$z(u,v) = -1$

$-1 \le u \le 1 \qquad -1 \le v \le 1$

II. $\operatorname{curl} \vec{F}(\vec{r}(u,v)) = \langle -u^2, uv + 2uv, v + u \rangle$

III. $\vec{r}_u = \langle 1, 0, 0 \rangle = \vec{i}$

$\vec{r}_v = \langle 0, 1, 0 \rangle = \vec{j}$

$\vec{r}_u \times \vec{r}_v = \vec{i} \times \vec{j} = \vec{k}$

IV $\iint \langle -u^2, uv + 2uv, v + u \rangle \cdot \langle 0, 0, 1 \rangle \, dA$

$= \int_{-1}^{1} \int_{-1}^{1} (v + u) \, du \, dv$

$= \int_{-1}^{1} \left(uv + \frac{u^2}{2}\right)\Big|_{-1}^{1} dv$

$= \int_{-1}^{1} \left[\left(v + \frac{1}{2}\right) - \left(-v + \frac{1}{2}\right)\right] dv$

$= \int_{-1}^{1} 2v \, dv = v^2 \Big|_{-1}^{1} = 1 - 1 = 0$

10. Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x,y) = \dfrac{-y\,\mathbf{i} + x\,\mathbf{j}}{x^2+y^2}$ and C consists of the right half of a unit circle centered at (1,0), then the line segment from (1,1) to (-1,1), then the left half of a unit circle centered at (-1,0), and then finally a line segment from (-1,-1) to (1,-1), all oriented counterclockwise.

Yikes! That's a crazy path! I think Green's would be nice <u>except</u> at (0,0)...

$$\frac{\partial Q}{\partial x} = \frac{1(x^2+y^2) - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial P}{\partial y} = \frac{-1(x^2+y^2) - -y(2y)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

So the integrand in Green's is zero, and thus no matter what the region is, the integral is zero, <u>except</u> if the region includes (0,0).

So let's separately try a small circle around the origin:

I. $\vec{r}(t) = \langle \cos t, \sin t \rangle$

II. $\vec{F}(\vec{r}(t)) = \left\langle \dfrac{-\sin t}{\sin^2 t + \cos^2 t}, \dfrac{\cos t}{\sin^2 t + \cos^2 t} \right\rangle = \langle -\sin t, \cos t \rangle$

III. $\vec{r}\,'(t) = \langle -\sin t, \cos t \rangle$

IV. $\displaystyle\int_0^{2\pi} (\sin^2 t + \cos^2 t)\,dt$

$\displaystyle = \int_0^{2\pi} 1\,dt$

$= 2\pi$

So by the long way, the integral around that inner circle is 2π, and by Green's, the integral over any other region is 0, so the total is 2π