Exam 1 **Real Analysis 1** **10/6/2006**

Each problem is worth 10 points. Show adequate justification for full credit. Don't panic.

1. State the definition of convergence of a sequence.

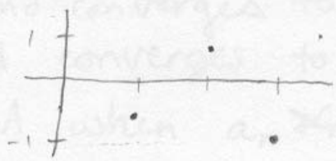
A sequence $\{a_n\}$ converges to A iff $\forall \varepsilon > 0\ \exists n^* \in \mathbb{N} \ni |a_n - A| < \varepsilon\ \forall n > n^*$.

2. State the definition of a decreasing sequence.

A sequence $\{a_n\}$ is decreasing iff $m > n$ then $a_m \leq a_n$

3. Give an example of a set with exactly two accumulation points.

$$\left\{ \frac{(-1)^n \cdot n}{n+1} \right\}$$



acc. points: -1 and 1

Excellent

4. Prove that for $f(x) = x^2$, $\lim_{x \to 5} f(x) = 25$.

Note 5 is an accumulation point for D_f

Let $\varepsilon > 0$ be given

Let $\delta = \min\{1, \frac{\varepsilon}{11}\}$

Then, when

$0 < |x-5| < \delta = \frac{\varepsilon}{11}$

we have

$11|x-5| < \varepsilon$

$|x+5||x-5| < \varepsilon$

$|x^2 - 25| < \varepsilon$

Great

Thus when $0 < |x-5| < \delta$ we have $|f(x) - 25| < \varepsilon$ as desired.

Scratch

$0 < |x-5| < \delta$

$|x^2 - 25| < \varepsilon$

$|x+5||x-5|$

if $\delta < 1$

$0 < x-5 < 1$

$5 < x < 6$

$10 < x+5 < 11$

$|x+5| < 11$

$11|x-5| < \varepsilon$

5. State and prove the Squeeze Theorem for sequences.

If a sequence $\{a_n\}$ converges to a limit L and another sequence $\{b_n\}$ converges to the same limit L, then if another sequence $\{c_n\}$ exists $\ni$ $a_n \leq c_n \leq b_n$ $\forall n \in \mathbb{N}$, then $\{c_n\}$ converges to L, also.

Since $\{a_n\}$ converges, let $\varepsilon > 0$

$\exists$ $n_1^* \in \mathbb{N} \ni \forall n > n_1^*$, $|a_n - L| < \varepsilon$ $\Rightarrow$ $-\varepsilon < a_n - L < \varepsilon$

$-\varepsilon + L < a_n < \varepsilon + L$ ①

also, for $\{b_n\}$

$\exists n_2^* \in \mathbb{N} \ni \forall n > n_2^*$, $|b_n - L| < \varepsilon$ $\Rightarrow$ $-\varepsilon < b_n - L < \varepsilon$

$-\varepsilon + L < b_n < \varepsilon + L$ ②

Using ① and ② and the assumption that $a_n \leq c_n \leq b_n$

$$-\varepsilon + L < a_n \leq c_n \leq b_n < \varepsilon + L$$

$$-\varepsilon + L < c_n < \varepsilon + L$$

$$-\varepsilon < c_n - L < \varepsilon$$

$$|c_n - L| < \varepsilon$$

Nice!

So $\forall n > n^* [= \max(n_1^*, n_2^*)]$, $|c_n - L| < \varepsilon$

By definition of convergence, c_n converges to L.

6. State and prove the Monotone Convergence Theorem (proof of *either* case is acceptable).

Def: A sequence $\{a_n\}$ that is monotone & bounded, converges.

proof: (assume increasing) Consider the set $S = \{a_n \mid n \in \mathbb{N}\}$. Since we know it's bounded then by the completeness axiom, there exists a least upper bound, L. Let $\varepsilon > 0$ be given. Then $\exists n^* \ni a_{n^*} > L - \varepsilon$. If $a_{n^*} < L - \varepsilon$ then $L - \varepsilon$ would be another bound. We also know that it's increasing, so $\forall n > n^*$, $a_n > a_{n^*}$. We also know $a_n < L + \varepsilon$ because L was a bound and no points can come after L. So let's combine all of these inequalities:

$L - \varepsilon < a_{n^*} < a_n < L + \varepsilon$. So by transitivity we can get rid of a_{n^*}.

$L - \varepsilon < a_n < L + \varepsilon$ OR $-\varepsilon < a_n - L < \varepsilon$ or $|a_n - L| < \varepsilon$

Which is the definition of converging. ∎

Well done!

7. Suppose that f and g are functions with both with domain $D \subseteq \mathbb{R}$. Prove that if $\lim_{x\to\infty} f(x) = A$ and $\lim_{x\to\infty} g(x) = B$ then $\lim_{x\to\infty} f \cdot g(x) = A \cdot B$.

Pf: By given, $\varepsilon > 0$. $|f(x)| < M$, for $M > 0$, $x \in D$.

$\lim_{x\to\infty} f(x) = A \Rightarrow |f(x) - A| < \boxed{\frac{\varepsilon}{2|B|+1}}$ for all $x \geq M_1$, $x \in D$

$\lim_{x\to\infty} g(x) = B \Rightarrow |g(x) - B| < \boxed{\frac{\varepsilon}{2|M|}}$ for all $x \geq M_2$ $x \in D$

If we have $M = \max(M_1, M_2)$ then

$$\begin{aligned} |f \cdot g(x) - AB| &= |f \cdot g(x) - f(x)B + f(x) \cdot B - AB| \\ &= |f(x)(g(x) - B) + B(f(x) - A)| \\ &\leq |f(x)||g x - B| + |B||(f(x) - A)| \\ &< |M| \frac{\varepsilon}{2|M|} + |B| \frac{\varepsilon}{2|B|+1} \\ &= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Good.

Actually, I think the proof will be similar to the proof of sequences.

True.

8. We have repeatedly used the proposition that any finite set of real numbers has a smallest element (one that's less than or equal to all the others). Prove this proposition.

Let's induct.

Base case: S'pose the set S has 1 element. Then we know that lone element $s \in S$ is less than or equal to all the other pts in S. $s \leq s$ so $s = s$.

S'pose there are $K \in \mathbb{Z}$ elements of our finite set, and let t be the smallest element in S. Prove it's true for $K+1$ elements in our finite set S. We are only adding one element to our set. So let's call it $m \in S$. All we need to do is see if $m \leq t$ (where t is our smallest element with K elements in S as stated in our Inductive Hypothesis). If $m \leq t$ then m is our new smallest element of S. If $m > t$, then we still have t being our smallest element of S.

So, if we have a finite set of real numbers, using induction, we can always find the smallest element. ∎

Excellent

9. Prove that if f and g are defined on (a, ∞) for some $a \in \mathbb{R}$, $\lim_{x\to\infty} f(x) = L$, and $\lim_{x\to\infty} g(x) = +\infty$, then $\lim_{x\to\infty} (f \circ g)(x) = L$.

Proof: Well, let $\varepsilon > 0$ be given. Then since $\lim_{x\to\infty} f(x) = L$, there must be an M for which $x > M$ implies $|f(x) - L| < \varepsilon$. But since $\lim_{x\to\infty} g(x) = +\infty$, there must be some N for which $x > N$ implies $g(x) > M$. Then we know that whenever $x > N$ we have $g(x) > M$, and in turn $g(x) > M$ implies $|f(g(x)) - L| < \varepsilon$, so $\lim_{x\to\infty} (f \circ g)(x) = L$ as desired. $\square$

10. Prove that no point outside the interval [0,1] is an accumulation point of [0,1].

Proof: Well, anything outside $[0,1]$ is either greater ~~than~~ 1 or less than 0.

Case 1: S'pose $s_0 > 1$ and s_0 is an accumulation point of $[0,1]$. But then $s_0 - 1 > 0$, so $\frac{s_0 - 1}{2} > 0$, and thus $\frac{s_0 - 1}{2} = \varepsilon$ is a value of ε which produces a deleted neighborhood of s_0 containing no points of $[0,1]$, contradicting the supposition that s_0 was an accumulation point.

Case 2: S'pose $s_0 < 0$ and s_0 is an accumulation point of $[0,1]$. But then $-s_0 > 0$ serves as an ε producing a deleted neighborhood of s_0 containing no points of $[0,1]$, again contradicting the supposition that s_0 was an accumulation point.

Thus in either case the supposition that an accumulation point outside $[0,1]$ led to a contradiction, as desired. $\square$