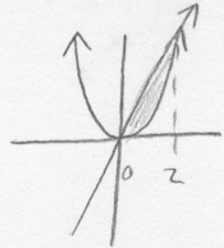
Exam 1 Calc 2 9/14/2007

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Write an integral for the area of the region between the graphs of $y = x^2$ and $y = 2x$.

$$\int_0^2 \left((2x)-(x^2)\right)dx$$

Good



$x^2 = 2x$

$x^2 - 2x = 0$

$(x)(x-2) = 0$

$x = 0, 2$

2. Write an integral for the average value of $f(x) = \sin x$ on the interval $[0, \pi]$.

$$\frac{1}{\pi - 0}\int_0^{\pi} \sin x \, dx \qquad \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$\frac{1}{\pi}\int_0^{\pi} \sin x \, dx$$

Great

3. Evaluate $\int \frac{1+x}{1+x^2}\,dx$.

$$\int \frac{1}{1+x^2} + \int \frac{x}{(1+x^2)}$$

$u = x^2+1$

$du = 2x\,dx$

$dx = \frac{du}{2x}$

$$\int \frac{\cancel{x}}{u} \frac{du}{2\cancel{x}}$$

$$\frac{1}{2}\int \frac{1}{u}\,du$$

$\tan^{-1} x$

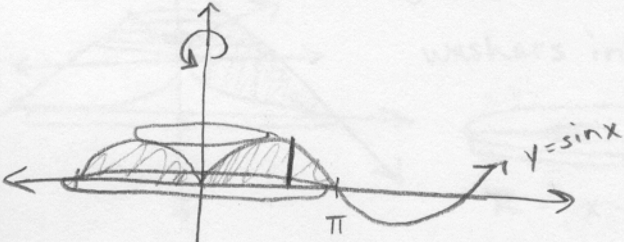
$$\frac{1}{2}\left[\ln|u|\right]$$

no ab. val. because always positive

$$\tan^{-1} x + \frac{1}{2}\left[\ln(x^2+1)\right] + C$$

Excellent!

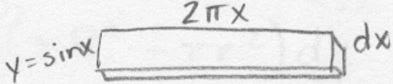
4. Write an integral for the volume of the solid obtained by rotating the region between the graph of $y = \sin x$ and the x-axis, for values of x between 0 and π, around the y-axis.



$$V = \int_0^{\pi} 2\pi x \cdot \sin x \, dx$$

$$V = 2\pi \int_0^{\pi} x \cdot \sin x \, dx$$

shells in dx



Nice.

5. A spring has a natural length of 50cm, and 80N hold it stretched to a length of 60 cm. How much work, to the nearest tenth of a Joule, is done in stretching the spring from a length of 60cm to a length of 70cm?

$W = F \cdot d$

natural length 50 cm → .5 m

$F = kx$

$60cm - 50cm = \frac{10 cm}{100} \rightarrow .1m$

$80N = k(.1m)$

$k = 800$

$70cm - 50cm = 20cm \cdot \frac{1m}{100cm}$

$.2m$

$W = \int_{.1}^{.2} 800x \, dx$

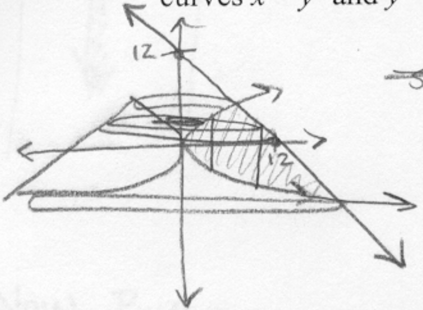
$F = 800x$

Great

$W = \left[\frac{800}{2} x^2\right]_{.1}^{.2} \rightarrow 400(.2)^2 - 400(.1)^2$

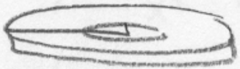
$\boxed{W = 12.0 J}$

6. Write an integral for the volume of the solid obtained by rotating the region bounded by the curves $x = y^2$ and $y = 12 - x$ around the y-axis.



~~shells in dx~~

washers in dy



dy

$y^2 = 12 - y$

$y^2 + y - 12$

$(y-3)(y+4) = 0$

$y = -4, 3$

R = x-value in terms of y = $12 - y = R$

r = x-value in terms of y

$r = y^2$

$$V = \int_{-4}^{3} (\pi R^2 - \pi r^2)\, dy$$

$$\boxed{V = \pi \int_{-4}^{3} \left((12-y)^2 - y^4\right) dy}$$

Good!

7. Bunny is a calculus student at Enormous State University, and she's having some trouble. Bunny says "Ohmygod. We had this quiz in Calc, and I bombed it – I'm *so* going to fail! I didn't even get, like, partial credit, which I think is totally unfair. It was this question about integrating, like 1 over x^2 from -2 to 3, and I triplechecked everything, but what I got was negative and the TA said that's automatically wrong and American students are terrible because they don't understand about checking assumptions or something. I think I'll drop."

Help Bunny by explaining to her what might be wrong with what she did.

This is a problem because in order to use the Fundamental Theorem of Calculus, the function must be continuous on the given interval. But this function isn't continuous, since there's an asymptote at $x=0$. So that means we don't have a way to determine the area under this curve now.

But even if she can't use the usual way of finding the area, Bunny should know that since the curve is above the x-axis, a negative value for the integral must be a sign of something wrong.

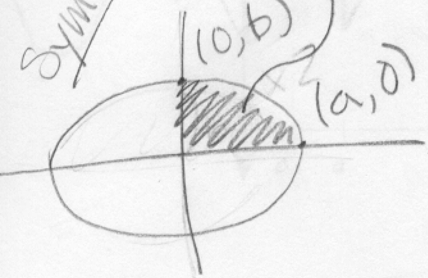
8. Set up an integral or integrals for the area inside the ellipse with equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$y^2 = b^2 - \frac{b^2x^2}{a^2}$$

$$y = \pm\sqrt{b^2 - \frac{b^2x^2}{a^2}}$$

Symmetry!



$$\boxed{A = 4\int_0^a \sqrt{b^2 - \frac{b^2x^2}{a^2}}\,dx}$$

Nice Job!

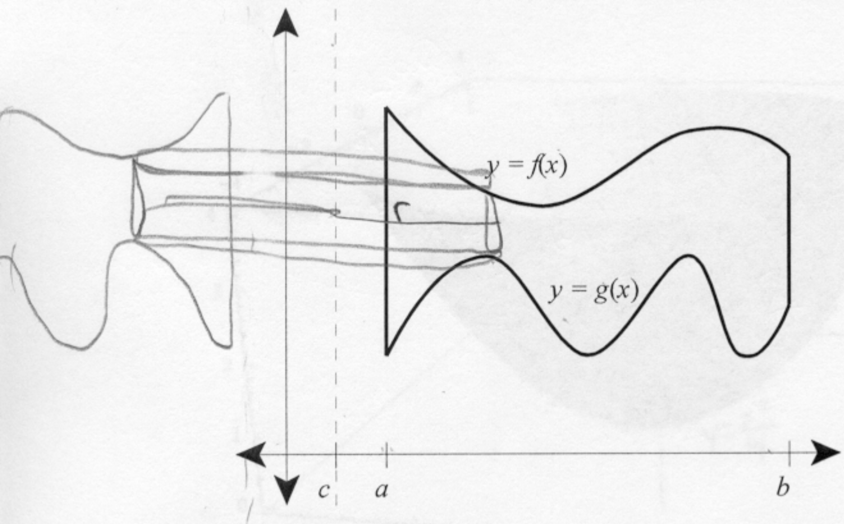
$$\frac{x^2}{a^2} + \frac{0^2}{b^2} = 1$$

$$\frac{x^2}{a^2} = 1$$

$$x^2 = a^2$$

$$x = a$$

9. Find a formula for the volume of the solid obtained by taking the region between $y = f(x)$ and $y = g(x)$, on the interval from $x = a$ to $x = b$, and rotating it around the axis $x = c$. You should assume $0 < c < a < b$ and that $f(x) > g(x)$ for all values of x between a and b, as shown.



$$\boxed{V = \int_a^b 2\pi(x-c)\big(f(x)-g(x)\big)\,dx}$$

Wonderful!

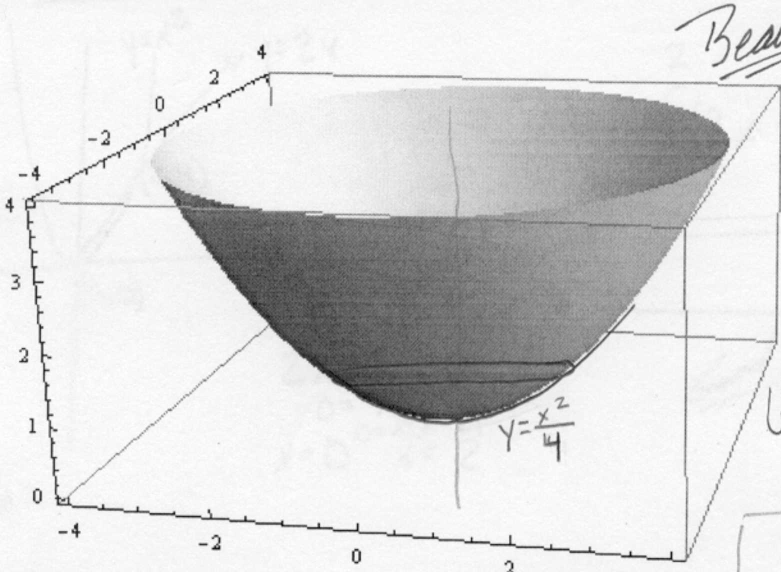
* radius $= (x - c)$

* height $= f(x) - g(x)$ top - bottom

* The graph goes from a to b, so those are the limits of integration.

Use cylindrical shells because we are rotating around $x=c$ which is similar to rotating around the y-axis. Same concept.

10. A tank full of water has the shape of the paraboloid of revolution formed by rotating the portion of $y = x^2/4$ below $y = 4$ around the y-axis, as shown below. Set up an integral expressing the work required to pump all of the water in this tank out over the top. [From Stewart 5th, Ch. 6 Review.]



Beautiful!

radius of a slice $= 2\sqrt{y}$ ft

area of a slice $= \pi(2\sqrt{y})^2$ ft^2

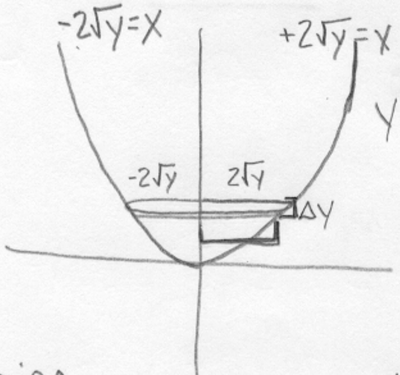
Volume of a slice $= \pi(2\sqrt{y})^2 \Delta y$ ft^3

Force of a slice $= 62.5\pi(2\sqrt{y})^2 \Delta y$ lbs

Work of a slice $= \overbrace{62.5\pi(2\sqrt{y})^2 \Delta y}^{\text{Force}} \cdot \overbrace{(4-y)}^{d}$ ft-lbs

$$\boxed{W = \int_0^4 62.5\pi(2\sqrt{y})^2 \cdot (4-y)\,dy}$$

Finding the radius



$y = \frac{x^2}{4}$

radius = x value

$4y = x^2$

$\pm\sqrt{4y} = x$

$\pm 2\sqrt{y} = x$

Imagine circular disks

y	d
0	4
4	0

$\frac{4-0}{0-4} = \frac{4}{-4} = -1$

$4 = -1(0) + b$

$4 = b$

$d = -1y + 4$

$d = 4 - y$

Finding the distance