Exam 1b Calc 3 9/30/2008

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. State the formal definition of the partial derivative of a function f(x, y) with respect to y.

$$f_y(x,y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x,y)}{h}$$

Good

2. Show that $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ does not exist.

*The limit does not approach the same value when y approaches from different directions so the limit does not exist.

Well, look at when $y=0$:

$$\lim_{(x,y)\to(0,0)} \frac{x(0)}{x^2+(0)^2} \to \lim_{x\to 0} 0 \to \underline{\underline{0}}$$

Now look at when $y=x$:

Great

$$\lim_{(x,y)\to(0,0)} \frac{x(x)}{x^2+(x)^2} \to \lim_{x\to 0} \frac{\cancel{x}^2}{2\cancel{x}^2} \to \underline{\underline{\frac{1}{2}}}$$

Since the limit approaches 2 different values when approaching from different directions, the limit does not exist.

3. Find the directional derivative of the function $g(x, y) = \ln(x^2 + y^2)$ at the point $(-2, 1)$ in the direction of the vector $\mathbf{v} = 5\mathbf{i} - 12\mathbf{j}$.

$\|\vec{v}\| = \sqrt{5^2 + 12^2}$

$= 13$

$\vec{v} = \langle 5, -12 \rangle$

$D_{\vec{u}} = \bar{a} \cdot g_x(x,y) + \bar{b}\, g_y(x,y)$

$\vec{u} = \langle \frac{5}{13}, -\frac{12}{13} \rangle$

$g_x = \dfrac{1}{(x^2+y^2)} \cdot 2x \qquad g_x(-2,1) = \dfrac{1}{5} \cdot -4 = -\frac{4}{5}$

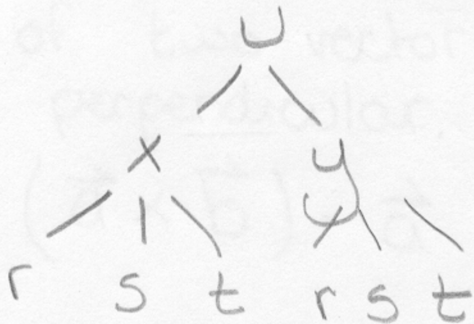
$g_y = \dfrac{1}{(x^2+y^2)} \cdot 2y \qquad g_y(-2,1) = \dfrac{1}{5} \cdot 2 = \frac{2}{5}$

$D_{\vec{u}} = \dfrac{5}{13} \cdot \dfrac{-4}{5} + \dfrac{-12}{13} \cdot \dfrac{2}{5} = \dfrac{-4}{13} + \dfrac{-24}{65}$

$= \dfrac{-20}{65} + \dfrac{-24}{65} = -\frac{44}{65}$

Excellent!

4. Write the appropriate version of the chain rule for $\frac{\partial u}{\partial t}$ in the case where $u = f(x, y)$, $x = x(r, s, t)$, and $y = y(r, s, t)$. Make clear distinction between derivatives and partial derivatives.



* all partial derivatives

Yes.

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial t}$$

5. Let $f(x,y) = x^3 + 3xy^2$. Find the direction in which f is increasing fastest at the point $(-3, 1)$, and the rate of increase in that direction.

$|\nabla f|$ = fastest rate of increase

∇f = direction where increase is greatest

Excellent!

$f(x,y) = x^3 + 3xy^2$

$f_x(x,y) = 3x^2 + 3y^2 \qquad f_x(-3,1) = 30$

$f_y(x,y) = 6xy \qquad f_y(-3,1) = -18$

$\nabla f = \langle 30, -18 \rangle \quad |\nabla f| = \sqrt{(30)^2 + (-18)^2} = \sqrt{1224} = 6\sqrt{34}$

$$\boxed{\begin{array}{l} \nabla f = \langle 30, -18 \rangle \\ |\nabla f| = 6\sqrt{34} \end{array}}$$

6. Show that for any vectors $\vec{a}$ and $\vec{b}$, the vector $\vec{a} \times \vec{b}$ is perpendicular to $\vec{a}$.

Well, we know that if the dot product of two vectors is 0, the two vectors are perpendicular. So, we need to show that $(\vec{a} \times \vec{b}) \cdot \vec{a} = 0$.

Let $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$.

Then $\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = a_2b_3\vec{i} + a_3b_1\vec{j} + a_1b_2\vec{k} - a_3b_2\vec{i} - a_1b_3\vec{j} - a_2b_1\vec{k}$

$= (a_2b_3 - a_3b_2)\vec{i} + (a_3b_1 - a_1b_3)\vec{j} + (a_1b_2 - a_2b_1)\vec{k}$

Excellent.

Then $\vec{a} \cdot (\vec{a} \times \vec{b}) = \langle a_1, a_2, a_3 \rangle \cdot \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$

$= \cancel{a_1a_2b_3} - \cancel{a_1a_3b_2} + \cancel{a_2a_3b_1} - \cancel{a_1a_2b_3} + \cancel{a_1a_3b_2} - \cancel{a_2a_3b_1}$

$= 0$

Since $\vec{a} \cdot (\vec{a} \times \vec{b}) = 0$, the vector $\vec{a} \times \vec{b}$ is $\perp$ to $\vec{a}$.

7. Find the maximum and minimum values (and distinguish clearly which are which) of the function $f(x,y) = x^2 - 4x + y^2$ subject to the constraint $x^2 + y^2 = 9$.

$f(x,y) = x^2 - 4x + y^2$ $\qquad x^2 + y^2 = 9$

$f_x(x,y) = 2x - 4$ $\qquad$ Lagrange multipliers $\qquad g(x,y) = k$

$f_y(x,y) = 2y$ $\qquad \nabla f(x,y) = \lambda \nabla g(x,y)$ $\qquad g_x(x,y) = 2x$

$g_y(x,y) = 2y$

$\langle 2x-4, 2y \rangle = \lambda \langle 2x, 2y \rangle$

$2x - 4 = 2x\lambda$

$2y = \lambda 2y \longrightarrow$ so $y = 0$ or $\lambda = 1$

$x^2 + y^2 = 9$

If $\lambda = 1$

$2x - 4 = 2x$

$-4 \neq 0$

so λ cannot $= 1$

If $y = 0$

$x^2 + 0 = 9$

$x = \pm 3$

Critical points are $(-3,0)$ and $(3,0)$

$f(-3,0) = (-3)^2 - 4(-3) + (0)^2$

$f(-3,0) = 9 + 12$

$= 21$

$f(3,0) = 3^2 - 4(3) + (0)^2$

$= 9 - 12$

$= -3$

$\boxed{\text{max value} = 21 \quad \text{min value} = -3}$

Well done!

8. Biff is a student taking calculus at Enormous State University and he's a bit confused. Biff says "Man, Calc 3 is killing me. This Lagraze stuff makes no sense at all. I'm good with solving the equations and everything, and I got the right points on the test question, but I did the second derivative test and it said they were all maxes, and so I picked that answer from the crazy multiple choices, and when I got the test back it was wrong. I doublechecked it, and the guy who sits next to me did the same thing, so we're pretty sure they got the answer key wrong, but the professor won't admit it and just says we need to read the hypotheses of the theorem, whatever that means!"

Help Biff out by explaining clearly why the second derivative test will or won't help him here, and how he should proceed.

Lagrange multipliers do not require the second derivative test to solve for the max and min values. When you solve the Lagrane multipliers, you obtain "important pts." not critical points. You aren't taking the first derivative of the partials and setting them equal to zero. It's a different process involving a scalar multiple. The gradient of the first function is equal to a scalar multiplied by the gradient of the restraint function. Once you solve the equations for the important points, you plug each pt into the original equation. The large value obtained is the maximum and the small value obtained is the minimum.

9. Find and classify all critical points of the function $f(x,y) = x^3 - y^3 - 2xy + 6$.

$f_x = 3x^2 - 2y$ $\qquad f_{xx} = 6x$

$f_{xy} = -2$

$f_y = -3y^2 - 2x$ $\qquad f_{yy} = -6y$

$0 = 3x^2 - 2y$ $\qquad 0 = -3y^2 - 2x$

$2y = 3x^2$ $\qquad 0 = -3\left(\frac{3}{2}x^2\right)^2 - 2x$

$y = \frac{3}{2}x^2$ $\qquad 0 = \frac{-27x^4}{4} - 2x$

$x = \frac{-2}{3}$ or $x = 0$

$\Downarrow \qquad \Downarrow$

$y = \frac{2}{3}$ $\qquad y = 0$

Well Done!

Critical points:

$(0,0)$ saddle pt

$\left(-\frac{2}{3}, \frac{2}{3}\right)$ maximum

$(0, 0, 6)$ saddle pt

$\left(-\frac{2}{3}, \frac{2}{3}, \frac{170}{27}\right)$ max

$D = f_{xx} \cdot f_{yy} - f_{xy}^2$

$D(0,0) = 6(0) \cdot -6(0) - (-2)^2 = -4 < 0$ saddle pt

$D\left(\frac{-2}{3}, \frac{2}{3}\right) = 6\left(\frac{-2}{3}\right) \cdot -6\left(\frac{2}{3}\right) - (-2)^2 = 12 > 0$ max

$f_{xx} < 0$

$6\left(-\frac{2}{3}\right) = -4$

10. Describe well the intersection of a sphere with radius 1 centered at the origin with a sphere of radius r centered at the point $(0,0,c)$.

Sphere w/ r = 1 at origin:

$$x^2+y^2+z^2=1^2$$

or

$$x^2+y^2=1-z^2$$

Sphere w/ r = r at $(0,0,c)$:

$$x^2+y^2+(z-c)^2=r^2$$

So they intersect where these coincide, or where

$$(1-z^2)+(z-c)^2=r^2$$

$$1-z^2+z^2-2zc+c^2=r^2$$

$$-2zc=r^2-1-c^2$$

$$z=\frac{r^2-1-c^2}{-2c}$$

$$z=\frac{c}{2}+\frac{1}{2c}-\frac{r^2}{2c}$$

And the intersection will be a circle centered on the z-axis in that plane, with radius we can obtain from:

$$x^2+y^2+\left(\frac{c}{2}+\frac{1}{2c}-\frac{r^2}{2c}\right)^2=1$$

$$x^2+y^2=1-\left(\frac{c}{2}+\frac{1}{2c}-\frac{r^2}{2c}\right)^2$$

So the radius is $\sqrt{1-\left(\frac{c}{2}+\frac{1}{2c}-\frac{r^2}{2c}\right)^2}$, as long as that value is real; otherwise they don't intersect.