Exam 1 Real Analysis 1 10/3/2008

Each problem is worth 10 points. Show adequate justification for full credit. Don't panic.

1. State the definition of the limit of a function $f(x)$ as x approaches $+\infty$.

Let $f(x)$ be a function $f: D \to \mathbb{R}$. and D is unbounded above. $f(x)$ has a limit as $x \to \infty$ if $\exists$ a real number L s.t. for any $\varepsilon > 0$ $\exists$ a real number $M > 0$ s.t.

$$|f(x) - L| < \varepsilon \quad \text{for } \forall x > M \text{ and } x \in D.$$

Good!

2. a) State the definition of an oscillatory sequence.

a sequence $\{a_n\}$ is oscillatory iff it does not converge, or diverge to $+\infty$ or $-\infty$.

Great

b) Give an example of an oscillatory sequence.

$a_n = (-1)^n$

3. a) Give an example of a function that converges to 5 as x approaches $+\infty$.

$f(x) = 5$:)

b) Give an example of a set with exactly two accumulation points.

Let $\{Z_n\}$ be defined $Z_n = \begin{cases} \frac{1}{n} & \text{if } n \text{ is even} \\ \frac{n}{n+1} & \text{if } n \text{ is odd} \end{cases}$

let S be the set $S = \{Z_n \mid n \in \mathbb{N}\}$ then S has two accumulation points.

Excellent!

4. State the Bolzano-Weierstrass Theorem for Sets.

Any infinite set that is bounded has at least one accumulation point.

Good!

5. Prove directly from the definition that $\lim_{x \to a} c \cdot x = c \cdot a$, where c is a real constant.

Def: $\forall \varepsilon > 0 \quad \exists \delta > 0$ s.t. $x \in D$ & $|x-a| < \delta$

$\Rightarrow |f(x) - L| < \varepsilon$

Let $\varepsilon > 0$ be given. Then let $\delta = \varepsilon / |c|$ where c is a real constant. From the def. we know $|x-a| < \delta$, so we have $|x-a| < \varepsilon/|c|$.

$\Rightarrow |c||x-a| < \varepsilon \Rightarrow |cx - ca| < \varepsilon$.

This means that $\lim_{x \to a} c \cdot x = c \cdot a$ since $\forall \varepsilon > 0 \quad \exists \delta = \varepsilon/|c|$ s.t. $x \in D$ & $|x-a| < \delta \Rightarrow |cx - ca| < \varepsilon$, or $|cf(x) - cL| < \varepsilon$, and the proof is complete.

We want:

$|cx - ca| < \varepsilon$

$|c||x-a| < \varepsilon$

$|x-a| < \varepsilon/|c|$

let $\delta = \varepsilon/|c|$

Well done!

6. Suppose that f and g are functions with both having domain $D \subseteq \mathbb{R}$. Prove that if $\lim_{x \to a} f(x) = A$ and $\lim_{x \to a} g(x) = B$ then $\lim_{x \to a}(f \cdot g)(x) = A \cdot B$.

Note that a must be an accumulation point

Since $\lim_{x \to a} g(x) = B$, fix $\varepsilon > 0$. Then $\exists\, \delta_1 > 0$ s.t

$0 < |x-a| < \delta_1$ and $x \in D \Rightarrow |g(x) - B| < \varepsilon$.

But $|g(x)| - |B| \le |g(x) - B|$, so $|g(x)| - |B| < \varepsilon$

or $|g(x)| < \varepsilon + |B|$ Let $\varepsilon + |B| = K$.

Now we also know $\exists\, \delta_2 > 0$ s.t

$0 < |x-a| < \delta_2$ & $x \in D \Rightarrow |f(x) - A| < \frac{\varepsilon}{2K+1}$

and $\exists\, \delta_3 > 0$ s.t

$0 < |x-a| < \delta_3$ & $x \in D \Rightarrow |g(x) - B| < \frac{\varepsilon}{2|A|+1}$

Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$ Then

$0 < |x-a| < \delta_3$ & $x \in D \Rightarrow$

Note $(f \cdot g)(x) = f(x)\, g(x)$

$|f(x) g(x) - AB| = |(f(x) g(x) - g(x) A) + g(x) A - AB|$

$\le |f(x) g(x) - g(x) A| + |g(x) A - AB|$ (triangle inequality)

$= |g(x)|\,|f(x) - A| + |A|\,|g(x) - B|$

10 $< K \frac{\varepsilon}{2K+1} + |A| \frac{\varepsilon}{2|A|+1} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

$\therefore \lim_{x \to a} (f \cdot g)(x) = A \cdot B$ $\square$

Nice Job!

7. State and prove the Monotone Convergence Theorem (proof of *either* case is acceptable).

State - If a sequence is monotone and bounded then it converges.

Proof: case when $\{a_n\}$ is increasing. Let $\varepsilon > 0$ be given.
Well we know $\{a_n\}$ is bounded so it must have a least upper bound, L, by the Completeness Axiom. This means that $L-\varepsilon$ is not be a least upper bound so $\exists n^* \in \mathbb{N}$ such that $a_{n^*} > L-\varepsilon$. Also since a_n is increasing we know $a_n > a_{n^*}$ $\forall n > n^*$. Additionally $a_n < L+\varepsilon$ since L is an upper bound so there can be no more points of a_n above it.

Well if we combine these inequalities we have

$$L-\varepsilon < a_{n^*} < a_n < L+\varepsilon \quad \forall n > n^*.$$

Well by transitivity

$$L-\varepsilon < a_n < L+\varepsilon \Rightarrow -\varepsilon < a_n - L < \varepsilon \Rightarrow |a_n - L| < \varepsilon.$$

$\forall n > n^*$. Hey this is the definition of converges. so since for $\forall \varepsilon > 0$ $\exists n^* \in \mathbb{N}$ such that $|a_n - L| < \varepsilon$ $\forall n > n^*$ we know a_n converges. $\square$

Nice!

8. Using some or all of the axioms:

(A1) (*Closure*) $a + b$, $a \cdot b \in \mathbb{R}$ for any $a, b \in \mathbb{R}$. Also, if $a, b, c, d \in \mathbb{R}$ with $a = b$ and $c = d$, then $a + c = b + d$ and $a \cdot c = b \cdot d$.

(A2) (*Commutative*) $a + b = b + a$ and $a \cdot b = b \cdot a$ for any $a, b \in \mathbb{R}$.

(A3) (*Associative*) $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for any $a, b, c, \in \mathbb{R}$.

(A4) (*Additive identity*) There exists a zero element in $\mathbb{R}$, denoted by 0, such that $a + 0 = a$ for any $a \in \mathbb{R}$.

(A5) (*Additive inverse*) For each $a \in \mathbb{R}$, there exists an element $-a$ in $\mathbb{R}$, such that $a + (-a) = 0$.

(A6) (*Multiplicative identity*) There exists an element in $\mathbb{R}$, which we denote by 1, such that $a \cdot 1 = a$ for any $a \in \mathbb{R}$.

(A7) (*Multiplicative inverse*) For each $a \in \mathbb{R}$ with $a \neq 0$, there exists an element in $\mathbb{R}$ denoted by $\frac{1}{a}$ or a^{-1}, such that $a \cdot a^{-1} = 1$.

(A8) (*Distributive*) $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ for any $a, b, c \in \mathbb{R}$.

(A9) (*Trichotomy*) For $a, b \in \mathbb{R}$, exactly one of the following is true: $a = b$, $a < b$, or $a > b$.

(A10) (*Transitive*) For $a, b \in \mathbb{R}$, if $a < b$ and $b < c$, then $a < c$.

(A11) For $a, b, c \in \mathbb{R}$, if $a < b$, then $a + c < b + c$.

(A12) For $a, b, c \in \mathbb{R}$, if $a < b$ and $c > 0$, then $ac < bc$.

Prove that if $a, b \in \mathbb{R}^+$, then $a < b$ **if and only if** $-a > -b$. Be explicit about which axioms you use.

$\Rightarrow$ As we know $a < b$

Note $-a$ and $-b$ exists [A5]

So $a < b$

$\Leftrightarrow a + (-a) < b + (-a)$ [A11].

$\Leftrightarrow 0 < b + (-a)$ [A5]

$\Leftrightarrow 0 + (-b) < (b + (-a)) + (-b)$ [A11]

$\Leftrightarrow 0 + (-b) < b + ((-a) + (-b))$ [A3]

$\Leftrightarrow (-b) + 0 < b + ((-b) + (-a))$ [A2].

$\Leftrightarrow (-b) + 0 < (b + (-b)) + (-a)$ [A3]

$\Leftrightarrow -b < (b + (-b)) + (-a)$ [A4]

$\Leftrightarrow -b < 0 + (-a)$ [A5].

$\Leftrightarrow -b < (-a) + 0$ [A2]

$\Leftrightarrow -b < -a$ [A4]

Very Nice

9. Show that if a sequence $\{a_n\}$ diverges to $-\infty$ and there exists some n_1 such that for all $n > n_1$ we have $a_n \geq b_n$, then the sequence $\{b_n\}$ must also diverge to $-\infty$.

let $M < 0$ be given

Since $\{a_n\} \to -\infty$ $\exists$ n^* S.T $a_n < M$ for $\forall n > n^*$

but $\exists$ n_1 S.T $a_n \geq b_n$ for $\forall$ $n > n_1$

well let $m = \max\{n_1, n^*\}$

and we get $b_n \leq a_n < M$ for all $n > m$

$\therefore$ $b_n < M$ for $\forall$ $M < 0$

Excellent!

So since $b_n < M$ for $\forall M < 0$ we have shown that $\{b_n\} \to -\infty$ also. * (negative of comparison Theorem.)

10. If the sequence $\{a_n\}$ converges to a nonzero constant A and $a_n \neq 0$ for any n, prove that the sequence $\left\{\frac{1}{a_n}\right\}$ is bounded.

Well, since $\{a_n\}$ converges to something other than zero and $\{1\}$ converges, then their quotient $\{\frac{1}{a_n}\}$ converges too. And since that converges, it's bounded by a theorem proved in class. $\square$