Exam 3 Calc 2 11/6/2009

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. a) Convert the point with rectangular coordinates (3,–3) to polar coordinates.

$\theta = \arctan\left(\frac{-3}{3}\right)$

$\theta = -\pi/4$

$x^2 + y^2 = r^2$

$9 + 9 = r^2$

$\sqrt{18} = r$

$(\sqrt{18}, -\pi/4)$

Excellent!

b) Convert the point with polar coordinates (2, π/2) to rectangular coordinates.

$x = r\cos\theta \quad x = 2\cos\pi/2 \quad x = 0$

$y = r\sin\theta \quad y = 2\sin\pi/2 \quad y = 2$

$(0, 2)$

Great!

2. Determine whether $y = x\,e^x$ is a solution to the differential equation $y'' - 2y' + y = 0$.

$$y' = (xe^x)' = \underline{e^x + xe^x}$$

$$y'' = (e^x)' + (xe^x)' = \underline{2e^x + xe^x}$$

$$y'' - 2y' + y = \underline{2e^x + xe^x} - 2\underline{(e^x + xe^x)} + \underline{xe^x}$$

= poof! poof! :)

$$= \underline{0}$$

$\rightarrow$ $y = xe^x$ is a solution.

Okay.

3. Consider the curve with parametric equations $x = 10 - t^2, y = t^3 - 12t$. Find the coordinates of all points on this curve where the tangent line is horizontal.

In order for the tangent line to be horizontal

$\frac{dy}{dx} = 0$

$y = t^3 - 12t$

$\frac{dy}{dx} = 3t^2 - 12$

$3t^2 - 12 = 0$

$t^2 - 4 = 0$

$t^2 = 4$

$t = \pm 2$

$x(2) = 10 - 4$
$= 6$

$y(2) = 8 - 24$
$= -16$

$x(-2) = 10 - 4$
$= 6$

$y(-2) = -8 + 24$
$= 16$

$\therefore$ The co-ordinates are $(6, -16)$ & $(6, 16)$

Excellent!

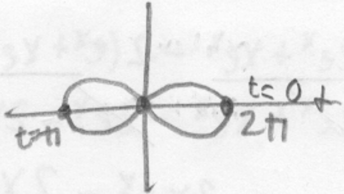
4. Set up an integral for the length of the curve with parametric equations $x = \cos t$, $y = \sin 2t$.

$$L = \int_{\alpha}^{\beta} \sqrt{(x')^2 + (y')^2}\, dt \qquad x' = -\sin t \qquad y' = 2\cos 2t$$

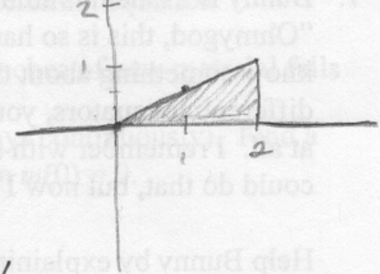
$$= \int_{0}^{2\pi} \sqrt{\sin^2 t + 4\cos^2 2t}\, dt$$

completes a full circ at $t = 2\pi$

Great Job!



5. Find the median of the p.d.f. $p(x) = \begin{cases} \frac{x}{2} & \text{if } 0 \le x \le 2 \\ 0 & \text{if } x<0 \text{ or } x>2 \end{cases}$.



$$\frac{1}{2} = \int_{-\infty}^{b} p(x)\,dx$$

break it apart

$$\frac{1}{2} = \int_{-\infty}^{0} 0\,dx + \int_{0}^{b} \frac{x}{2}\,dx \qquad \frac{1}{2} = \frac{1}{2}\int_{0}^{b} x\,dx$$

Excellent!

$$1 = \left[\frac{1}{2}x^2\right]_0^b \quad 1 = \left[\frac{1}{2}b^2\right] - \left[\frac{1}{2}(0)^2\right] \quad 1 = \frac{1}{2}b^2 \quad b = \pm\sqrt{2}$$

The value for b must be between 0 and 2 so $-\sqrt{2}$ is an extraneous solution. The median of $p(x)$ is $\boxed{\sqrt{2}}$

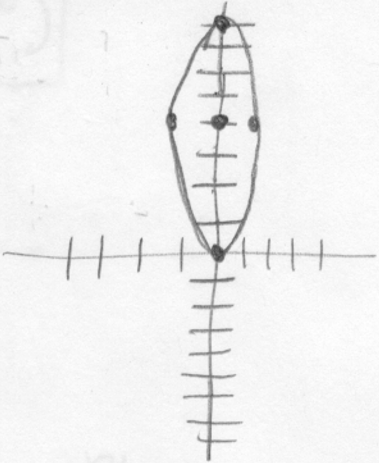
6. Write the conic section with equation $16x^2 + y^2 - 8y = 0$ in standard form, identify it as a parabola, hyperbola, or ellipse, and give coordinates of its vertices.

$16x^2 + (y^2 - 8y + 16) = 0 + 16$

$16x^2 + (y-4)^2 = 16$

$$\boxed{x^2 + \frac{(y-4)^2}{16} = 1}$$

ellipse



Excellent!

center: (0, 4)

vertices: (0,0)(0,8)(-1,4)(1,4)

7. Bunny is a calculus student at Enormous State University, and she has a question. Bunny says "Ohmygod, this is so hard. Our TA gave us, like, this hint? And it was that we needed to know something about the error if you use the Oiler's method, like for approximating differential equators, you know? But I looked in the book and there aren't any formulas for it at all. I remember with stuff like the midpoint approximations there were formulas, and I could do that, but now I'm totally panicking!"

Help Bunny by explaining what's most important to know about the errors involved in using Euler's method.

Well, its importaint to know that Oiler method isn't perfect, unless you are in very artifical circumstances your answer will be off. If the rate of change is trending upward then you are going to wind up with an underestamite. If the rate of change is trending downward then you will have an overestimate. Oiler method works by generating a bunch of straight lines that have the slope of the tangent line of the curve at that point if the curve had passed through it. The longer the lines are the less accurate your answer will be unless happenstance has it line up for some odd reason which isn't really that important. Anyway to avoid as much error as possible use very small lines/interva

Excellent Answer!

8. The buildup of mulch on a forest floor can be modeled by the differential equation $\frac{dm}{dt} = 2 - 0.4m$, (where m is the amount of mulch at time t, 2 inches of new material falls to the forest floor each year, and 40% of the material present decays continuously). Find a solution to this differential equation satisfying the initial condition $m(0) = 0$.

$$\int \frac{1}{2-0.4m}\,dm = \int 1\,dt$$

$$-\frac{1}{0.4}\ln|2-0.4m| = t + C \quad \text{constant}$$

$$\ln|2-0.4m| = -\frac{2}{5}t + C$$

$$|2-0.4m| = e^{\frac{-2t}{5}+C} = e^{-2t/5}\cdot \boxed{e^{C}}$$

constant which will absorb the absolute value signe

Yes!

$$2-0.4m = Ce^{-2t/5}$$

$$0.4m = 2 - Ce^{-2t/5}$$

$$m = \frac{5}{2}\left[2 - Ce^{-2t/5}\right]$$

$$m = 5 - \frac{5}{2}Ce^{-2t/5}$$

make it satisfy $m(0)=0$

Nice Work

$$0 = 5 - \frac{5}{2}Ce^{0} = 5 - \frac{5}{2}C$$

$C = 2$ so...

$$\boxed{m(t) = 5 - 5e^{-2t/5}}$$

9. Find the area inside both $r = 1 + \cos\theta$ and $r = 1 - \sin\theta$.

Where do they intersect?

$1+\cos\theta = 1-\sin\theta$

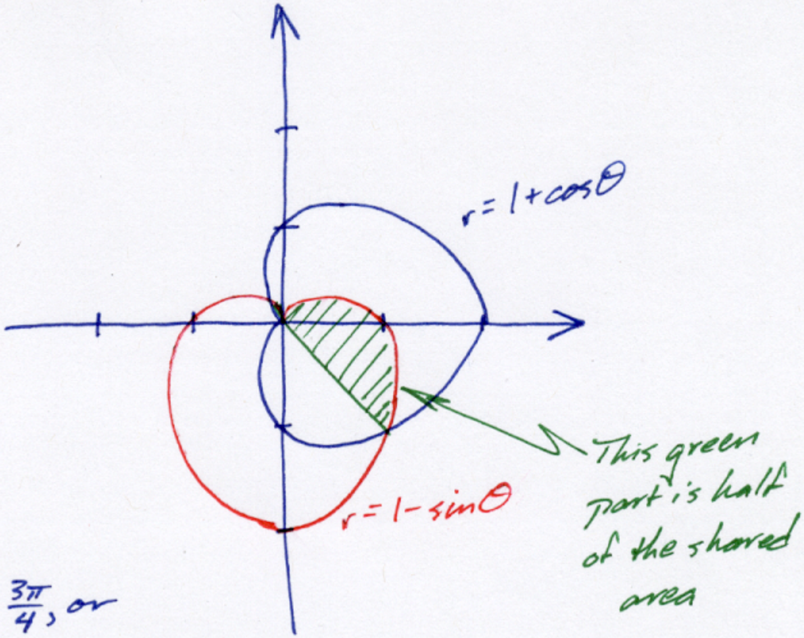
$\cos\theta = -\sin\theta$

$1 = -\frac{\sin\theta}{\cos\theta}$

$-1 = \tan\theta$

$\theta = \arctan -1$

$\theta = \frac{-\pi}{4}$ or $\frac{3\pi}{4}$



So using symmetry, we can double the area inside $r = 1-\sin\theta$ from $\theta = \frac{-\pi}{4}$ to $\theta = \frac{3\pi}{4}$, or

$$\text{Area} = 2\int_{-\pi/4}^{3\pi/4} \frac{1}{2}(1-\sin\theta)^2\,d\theta$$

$$= \int_{-\pi/4}^{3\pi/4} (1 - 2\sin\theta + \sin^2\theta)\,d\theta$$

$$= \left[\theta + 2\cos\theta + \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]_{-\pi/4}^{3\pi/4}$$

$$= \left(\frac{3\pi}{4} + 2\cdot\frac{-\sqrt{2}}{2} + \frac{1}{2}\cdot\frac{3\pi}{4} - \frac{1}{4}\cdot -1\right) - \left(\frac{-\pi}{4} + 2\cdot\frac{\sqrt{2}}{2} + \frac{1}{2}\cdot\frac{-\pi}{4} - \frac{1}{4}\cdot -1\right)$$

$$= \frac{9\pi}{8} - \sqrt{2} + \frac{1}{4} + \frac{3\pi}{8} - \sqrt{2} + \frac{-1}{4}$$

$$= \frac{3\pi}{2} - 2\sqrt{2}$$