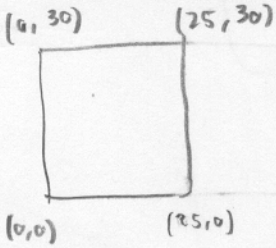
Exam 2 **Calc 3** **10/23/2009**

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Linn County is a rectangle approximately 25 miles from west to east and 30 miles from south to north. Suppose that the number of inches of rainfall in Linn County on a particular day is approximately given by $R(x, y) = 0.423 + 0.019x - 0.007y$, where x measures the distance in miles eastward from the western border of the county and y measures the distance in miles northward from the southern border of the county. Set up a double integral for the total rainfall in the county on that day.

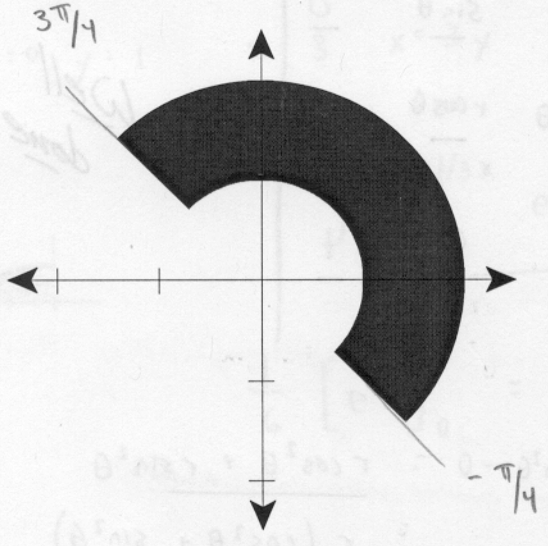


$$\int_0^{25}\int_0^{30} (0.423 + 0.019x - 0.007y)\, dy\, dx$$

Good!

2. Set up a double integral for the total mass of an object with density given by $d(x,y) = 2x + 5$ and shaped as shown below (where the tick marks on the axes represent units).

$2r\cos\theta + 5$

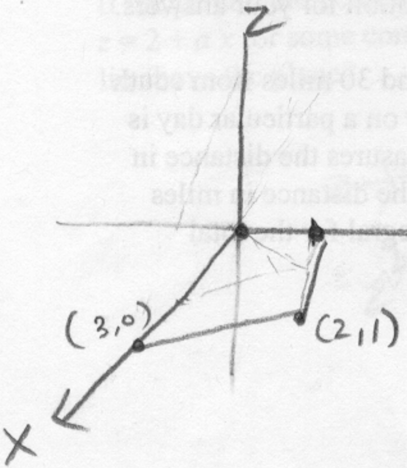


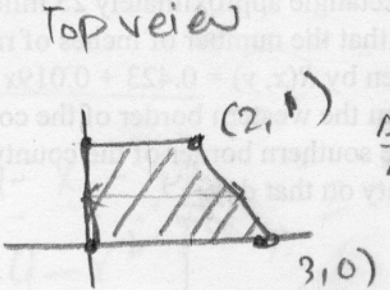
$$\int_{-\pi/4}^{3\pi/4}\int_1^2 (2x+5)\, r\, dr\, d\theta$$

$$\int_{-\pi/4}^{3\pi/4}\int_1^2 (2r^2\cos\theta + 5r)\, dr\, d\theta$$

Great!

3. Set up an iterated integral for the volume of the region in the first octant above the trapezoid with vertices (0,0), (3,0), (2,1), and (0,1) and below $z = 20 + x$.





$$\frac{\Delta y}{\Delta x} = \frac{1-0}{2-3} = \frac{1}{-1} = -1$$

So, $(y-0) = -1(x-3)$

$y = -x + 3$

$\boxed{x = 3 - y}$

$$\int_0^1 \int_0^{3-y} \int_{z=0}^{z=20+x} 1 \, dz \, dx \, dy$$

Nicely done.

4. Compute the Jacobian for the transformation $x = r\cos\theta, y = r\sin\theta, z = z$.

$$\text{Jacobian} = \begin{vmatrix} \frac{dx}{dr} & \frac{dy}{dr} & \frac{dz}{dr} \\ \frac{dx}{d\theta} & \frac{dy}{d\theta} & \frac{dz}{d\theta} \\ \frac{dx}{dz} & \frac{dy}{dz} & \frac{dz}{dz} \end{vmatrix} = \begin{vmatrix} \underline{\cos\theta} & \underline{\sin\theta} & \underline{0} \\ \underline{-r\sin\theta} & \underline{r\cos\theta} & \underline{0} \\ \underline{0} & \underline{0} & \underline{1} \end{vmatrix}$$

Well done

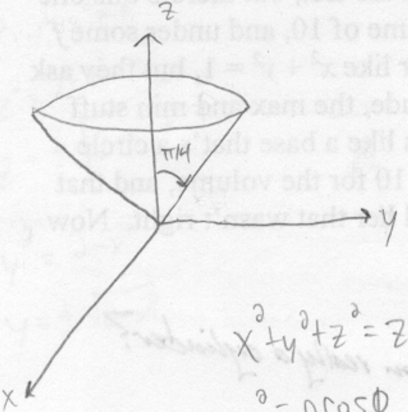
$$r\cos^2\theta + 0 + 0 - 0 - -r\sin^2\theta - 0 = \underline{r\cos^2\theta + r\sin^2\theta}$$

$$= \underline{r(\cos^2\theta + \sin^2\theta)}$$

$$= r(1)$$

$$= \boxed{r}$$

5. Set up an iterated integral for the volume of the solid that lies above the cone $z=\sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2=z$.



$$\int_0^{\pi/4}\int_0^{2\pi}\int_0^{\cos\phi} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$$x^2+y^2+z^2=z$$

$$\rho^2=\rho\cos\phi$$

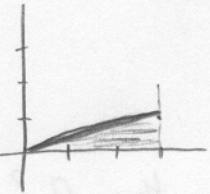
$$\rho=\cos\phi$$

Great

6. Evaluate $\int_0^1 \int_{3y}^3 e^{x^2}\, dx\, dy$.

need to <u>switch order</u> to $dy\,dx$

$y=0 \quad y=1 \qquad x=3 \quad x=3y \qquad y=\frac{1}{3}x$



$$\underline{\int_0^3 \int_0^{1/3 x} e^{x^2}\, dy\, dx} = \int_0^3 \left[y e^{x^2} \right]_0^{1/3 x} dx = \frac{1}{3}\int_0^3 x e^{x^2}\, dx$$

$u = x^2 \quad du = 2x \cdot dx$

$$\frac{1}{6}\left[e^{x^2}\right]_0^3 = \boxed{\frac{1}{6}\left(e^9 - 1\right)}$$

Excellent!

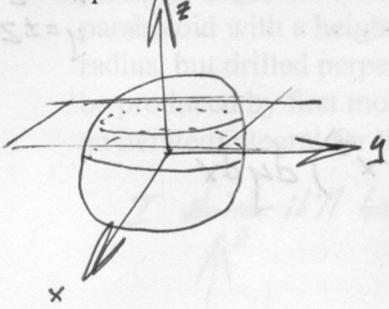
7. Biff is a calculus student at Enormous State University, and he's having some trouble. Biff says "Dude, I used to think multiple choice exams were totally easy, but my Calc class is just evil. They even give us some sample questions that might be on the test, but there's this one where I got no idea. It's like, they tell you something has a volume of 10, and under some f but they don't give you a formula for it, and it's inside a cylinder like $x^2 + y^2 = 1$, but they ask what's the maximum value f could have inside that cylinder. Dude, the max and min stuff was last test, so it's unfair so many ways. Anyway, I figured it's like a base that's a circle with area π, and so if the function was height $10/\pi$ then you get 10 for the volume, and that was one of the answers, but this girl in my class said the TA told her that wasn't right. Now what?"

Explain clearly to Biff how he might approach such a problem

Biff, you computed the average value that a function f such that $\int_0^{2\pi}\int_0^1 f\, r\, dr\, d\theta = 10$ must have, but the question asked for a maximum value. So if f has a value of less than $\frac{10}{\pi}$ over some portion of the circle, it can have a value of greater than $\frac{10}{\pi}$ over some other portion. Actually, the maximum value that f can have in the cylinder is unbounded, assuming you have explained the question right. For f can be zero everywhere exept a small region inside the unit circle, and in that region have some very large value such that the volume of this very thin column is 10.

Wonderful!

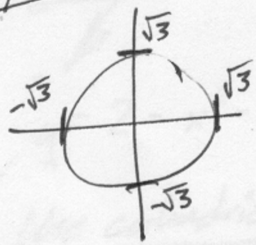
8. Set up iterated integrals for the x-coordinate of the center of mass of the region inside a sphere with radius 2 centered at the origin and above the plane $z = 1$.



$$\bar{x} = \frac{\displaystyle\int_{-\sqrt{3}}^{\sqrt{3}} \int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}} \int_{1}^{\sqrt{4-x^2-y^2}} x \cdot k \, dz\, dy\, dx}{\displaystyle\int_{-\sqrt{3}}^{\sqrt{3}} \int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}} \int_{1}^{\sqrt{4-x^2-y^2}} k \, dz\, dy\, dx}$$

(Assuming constant density k, since nobody said otherwise.)

Top View:



$x^2+y^2+z^2 = 2^2$

intersects $z=1$ along

$x^2+y^2+(1)^2 = 4$

$x^2+y^2 = 3$

a circle with radius $\sqrt{3}$

or$_1$ $$\bar{x} = \frac{\displaystyle\int_0^{2\pi} \int_0^{\sqrt{3}} \int_1^{\sqrt{4-r^2}} r\cos\theta \cdot k r \, dz\, dr\, d\theta}{\displaystyle\int_0^{2\pi} \int_0^{\sqrt{3}} \int_1^{\sqrt{4-r^2}} k r \, dz\, dr\, d\theta}$$

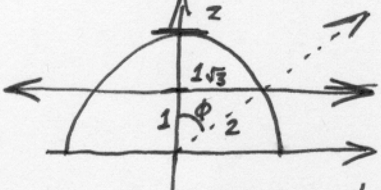
or$_2$ $$\bar{x} = \frac{\displaystyle\int_0^{2\pi} \int_0^{\pi/3} \int_{\frac{1}{\cos\phi}}^{2} \rho\sin\phi\cos\theta \cdot k \cdot \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta}{\displaystyle\int_0^{2\pi} \int_0^{\pi/3} \int_{\frac{1}{\cos\phi}}^{2} k \cdot \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta}$$

For spherical:

$z=1 \rightarrow \rho\cos\phi = 1$

$\rho = \frac{1}{\cos\phi}$

To find ϕ interval, look at cross section in xz-plane:



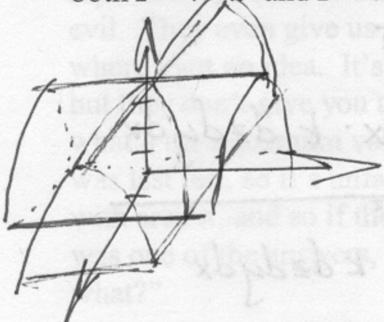
So $\cos\phi = \frac{1}{2}$

$\Rightarrow \phi = \pi/3$

9. Set up an iterated integral or integrals for the volume of the region above $z = 0$, but below both $z = 4 - x^2$ and $z = 4 - y^2$.

$z = 4 - x^2$ crosses $z = 0$ where $0 = 4 - x^2 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$

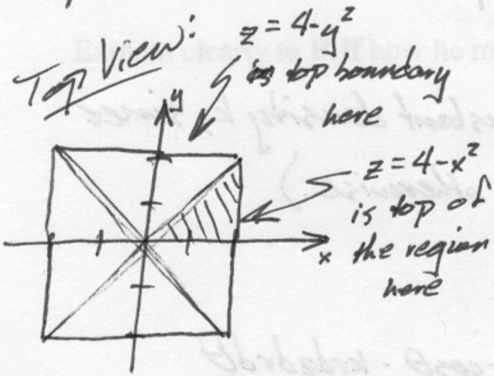
$z = 4 - y^2$ crosses $z = 0$ where . . . . . . $y = \pm 2$



$$\text{Volume} = 8 \cdot \int_0^2 \int_0^x (4 - x^2)\, dy\, dx$$

8 because of symmetry

Top View:



The surfaces cross where

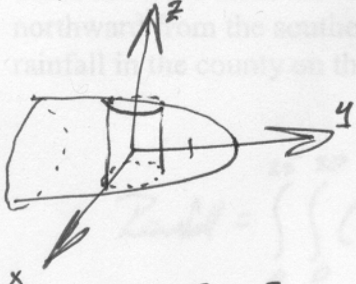
$$4 - x^2 = 4 - y^2$$

$$x^2 = y^2$$

$$x = \pm y$$

10. Jon has decided that current economic conditions are ideal for starting a business that produces mathematically perfect beads. His first product will be beads shaped like a paraboloid with a height of 4mm and circular base of radius 2mm. The hole will be 1mm in radius, but drilled perpendicular to the axis of symmetry of the paraboloid. These beads will be produced by first molding the complete paraboloid, and then drilling out the hole. Set up an iterated integral for the volume of material that will be removed in making the hole.

I think it'll be easiest if I put the hole centered on the z-axis:



$$\text{Volume} = \int_0^{2\pi} \int_0^1 \int_{-\sqrt{2-x^2-y}}^{\sqrt{2-x^2-y}} 1 \cdot r \, dz \, dr \, d\theta$$

$y = 2 - x^2 - z^2$

Use cylindrical!

$z^2 = 2 - x^2 - y$

$z = \pm\sqrt{2 - x^2 - y}$