L-Tromino Tiling of Mutilated Chessboards

Martin Gardner

Apart from compilations of his Mathematical Games columns, more than 70 other books **Martin Gardner** has written concern science, philosophy, and literature, the best known being *The Annotated Alice*, a study of Lewis Carroll's two books about Alice. *The Flight of Peter Fromm*, and *Visitors from Oz*, are novels. Gardner has also edited several anthologies of popular verse.

Gardner's lifelong hobby is conjuring, about which he has written books for the trade. As his good friend Persi Diaconis has written, "Warning: Martin Gardner has turned hundreds of mathematicians into magicians and hundreds of magicians into mathematicians."

Gardner currently lives in Norman, Oklahoma and is as active writing as ever.

Suppose a standard chessboard is 'mutilated' by the removal of two diagonally opposite corner cells. Can the remaining 62 squares be tiled with 31 dominos? The answer is 'no' because the removed squares are the *same* color. Say the color is white. The remaining 62 squares will have an excess of two black cells. Each domino covers one black and one white cell. After 30 are placed, two black cells will remain uncovered. They cannot be adjacent, therefore they can't be covered by a domino. This famous puzzle, solved by a simple parity check, is a simple example of a tiling problem on a mutilated chess board.

Less well known is the following related problem. Assume the chessboard is mutilated by having two cells removed of *opposite* color from anywhere on the board. Can the remaining 62 squares always be tiled by dominoes? The answer is yes, and there is a lovely proof by Ralph Gomory [**2**].

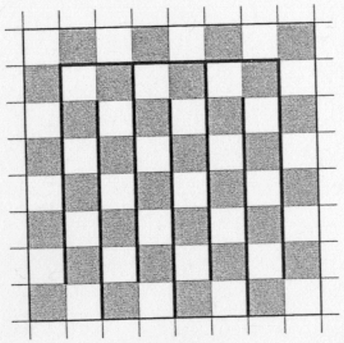

Figure 1. Gomory's proof.

Imagine heavy lines drawn on the chessboard as shown in Figure 1. They outline a closed path along which the squares are like beads of alternating color on a necklace.

If any two cells of opposite color are taken from the path, it will cut the path into two open-ended segments, or one segment if the removed cells are adjacent. Each segment will consist of an even number of cells of alternating colors, therefore it can be tiled with dominoes. Gomory's clever proof is easily generalized to all square boards with an even number of cells.

If, instead of dominos, we tile with L-trominos, also called bent, or V, or right trominos, then all square boards with a number of cells divisible by 3 can be tiled except for the 3×3 board. We will not be concerned with such 'whole' boards, but only with mutilated boards with a number of cells that is a multiple of 3 after a single cell has been removed from any spot on the board. We will call such boards *deficient*. In other words, a board of side n is deficient if $n^2 - 1$ is a multiple of 3, i.e., n is *not* a multiple of 3. The sides of such boards form the sequence

$$2, 4, 5, 7, 8, 10, 11, 13, 14, \ldots \qquad (*)$$

We will call these numbers the *orders* of a board and, from now on, the word *tromino* will mean an L-tromino exclusively.

Our basic question is this: What deficient boards with sides in the sequence $(*)$ can be tiled without gaps or overlaps with L-trominos after a cell has been taken from anywhere on the board? We will take up these boards roughly in numerical order, culminating with a statement of the complete solution.

Powers of 2

Consider the order-2 board first. It obviously is tilable with any cell missing (see Figure 2, left). Figure 2, right, shows how the order-4 can be tiled. The 2×2 square takes care of a missing cell in each of its four corners. The rest of the board is tiled by taking advantage of what Solomon Golomb named a rep-tile—a tile that can form an enlarged replica of itself. The top left 2×2 square rotates to put its missing cell in four places, and the entire order-4 square rotates to carry the missing cell to any of its sixteen places.

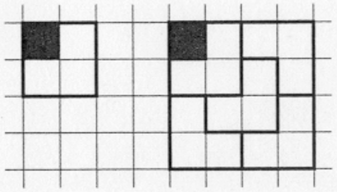

Figure 2. Orders 2 and 4.

In 1953 Golomb, the "father" of polyominoes (he named them and was the first to study them in depth) discovered a beautiful proof by induction that all boards with sides in the doubling sequence 2, 4, 8, 16, ... could be tiled with trominoes when any cell is missing. The proof was first published in [**3**]. It is repeated on pages 27–28 of [**4**]. Numerous mathematicians have since included the proof in their books, often without credit to Golomb. Roger Nelsen, in [**6**], gives Golomb's proof with a wordless single diagram.

Golomb's famous proof starts with the 2×2 case shown on the left of Figure 3. This square is placed in the corner of the order-4 as shown at the center of Figure 3. The 4×4 then goes in the corner of an order-8 (shown on the right) and a tromino placed at the corner of the shaded order-4. We know the dark square can be tiled with any cell missing, and we know the three unshaded quadrants can be tiled with trominoes because each has a missing corner cell. By rotating the board, a missing cell at any spot in the shaded quadrant can be brought to any spot on the order-8 board.

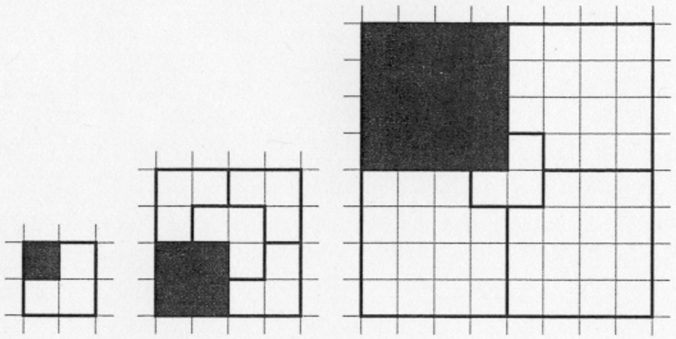

Figure 3. Golomb's induction proof.

Orders 5 and 7

The order-5 board is next, as 5 is the next unsolved number in the sequence ($*$). It has a neat symmetrical tiling when the center cell is gone, as shown in Figure 4, left. I have tiled this board with four 2×3 tiles. Each is tilable with two trominoes in two different ways. Using 2×3 tiles is a valuable device for solving tromino problems.

When the missing cell is the one shown black in Figure 4, center, the cell above it must be covered by a tromino on either side. In each case, shown here with a tromino above and on the right, this produces two cells (numbered 1 and 2) that cannot be covered with a tile. Indeed, the order-5 square can be tiled only when the missing cell is one of the nine shown in black in Figure 4 right. As a pleasant exercise, see if you can tile the board when the missing cell is at a corner.

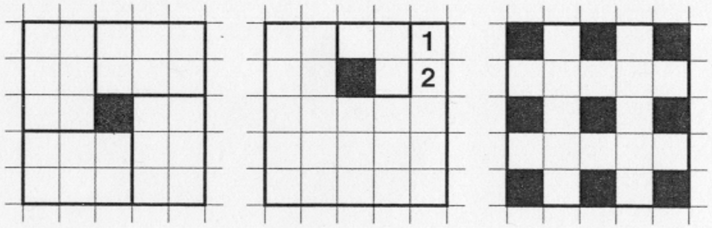

Figure 4. The order-5 square.

The order 7 board is more difficult to analyze. I was unable to find a single diagram that would prove this board tilable, but Golomb sent me his unpublished way of proving tilability with the aid of three diagrams.

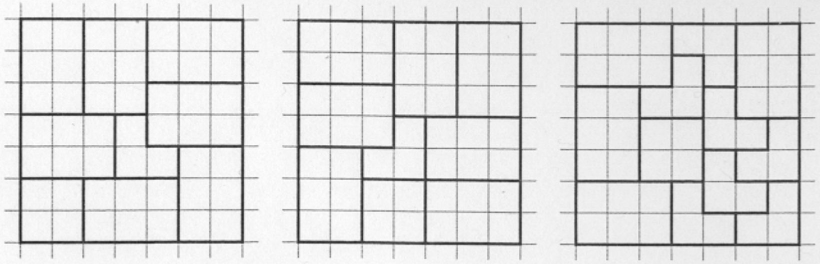

Figure 5. Golomb's proof that order-7 is tilable.

His proof goes like this. Figure 5 shows three tilings of the order-7 board. In each tiling, the 2×2 square obviously can be tiled with a tromino so that the missing cell is at any of the four corners. By rotating the three patterns, the missing cell can be placed at any spot on the board.

Somewhat more difficult is to find tilings that maximize the number of 2×3 tiles. As a challenge, can the reader find a tiling of the 7×7 board using six 2×3 tiles and 4 trominoes (see Figure 6)? The solution is unique except for a single reflection (see page 226).

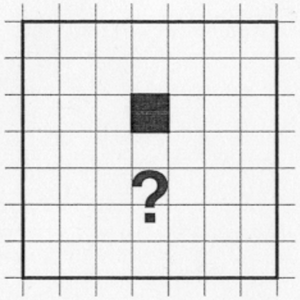

Figure 6. A challenge to the reader.

Note in Figure 5, that in each pattern the number of free trominoes–trominoes not in any 2×3 tile—is always even. This is no coincidence. It led me to the following trivial little law. When a board's order is even, the number of free trominoes in a tiling pattern is odd, and vice versa. When the board's number is odd, the number of free trominoes must be even.

The parity proof is simple. If a board's order is even, after a cell is removed there will be $(n^2 - 1)/3$ trominoes in any tiling, an odd number. Each 2×3 tile contains two trominoes, so the total number of trominoes in 2×3 tiles will be even. Subtracting this number from the odd total of trominoes and you get an odd number of trominoes not in any 2×3 tiles.

Suppose the board's order is odd. After a cell is removed there will remain an even number of cells. Subtracting the even number of trominoes in the 2×3 tiles leaves an even number of trominoes not in a 2×3 tile.

Beyond 7

Golomb's induction proof can be applied to an infinity of other doubling sequences. In particular, now that we have tiled the 7×7 board, we can tile boards of size $n \times n$ where n is of the form $2^k 7$. For example, consider the order-14 board. Divide it into quadrants with a shaded order-7 board in the top left corner, and attach a tromino to its lower right corner as before. Because the 7-board is tilable, the proof for order-14 follows, and of course leads by induction to proofs for orders 28, 56, 112, ...

A similar proof for the order-10 board can't be obtained by placing an order-5 in the corner because order-5 is not tilable, but we can handle it in a slightly different way. Put in the top left corner an order-8 which we know is tilable. The remaining area forms a path of width 2 along the bottom and right sides of the large square (see Figure 7). By rotations and reflections, each missing cell in the order-8 can be transferred to any cell on the board. This leads to proofs for orders 20, 40, 80, and so on. A similar proof for order-11 has an order-7 square in the corner, and a path of width 4 along bottom and side. It leads by induction to solutions for orders 22, 44, 88, ... Clearly this technique provides an infinity of doubling sequences for tilable boards. Simply, put in the top left corner of any board a tilable board with a side equal to or smaller than the larger board. If you can tile the path it leaves at the bottom and side, then the board is tilable.

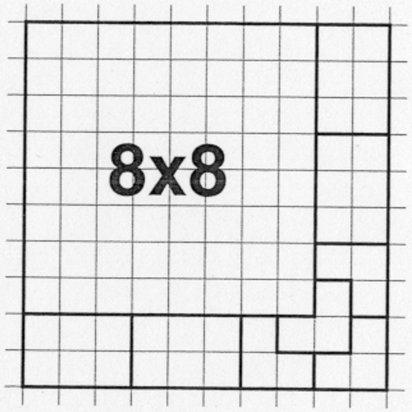

Figure 7. Proof that order-10 is tilable.

Boards with sides that are primes are usually the hardest to tile. Order 17 is solved by a corner square of side 13 and a path of width 4. Order 19 is solved by a corner square of order 14, in turn based on order-7, and a path of width 5. (See Figure 8.)

The complete result

By working with these patterns I came close, but not close enough, to finding an induction proof that all deficient squares are tilable except for order-5. A proof was finally obtained by I. Ping Chu and Richard Johnsonbaugh [**1**].

Chu and Johnsonbaugh not only took care of all deficient squares, but also all deficient rectangles! Their induction proof is too technical to repeat here. To summarize,

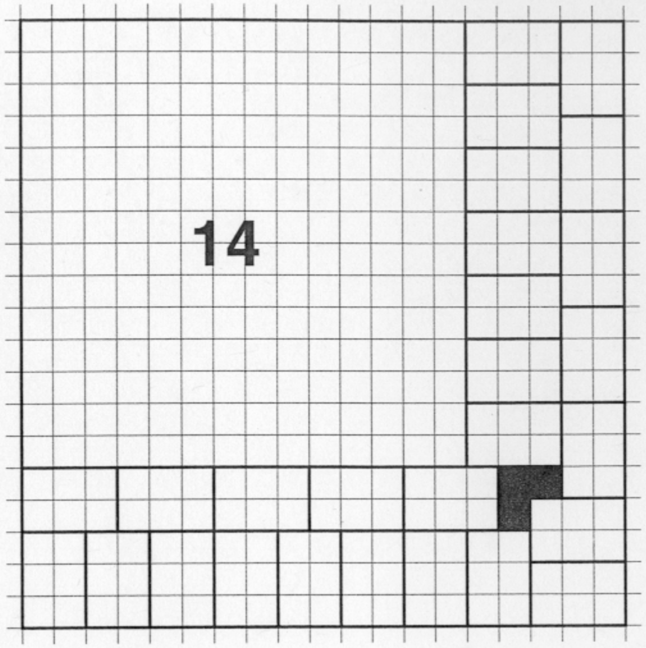

Figure 8. Order-19 is tilable.

they showed tilability for all m by n rectangles (including squares when $m = n$) which have a number of cells that is a multiple of 3 after a cell is removed. Such boards are tilable if and only if all of the following are true:

1. m is equal to or greater than 2.
2. n is equal to or greater than m.
3. if m is 2, n must be 2,
4. m is not 5.

A 4×7 rectangle is the smallest deficient rectangle, not a square, that is tilable with L-trominoes. As another exercise, see how long it takes you to tile it when the missing cell is at a corner, and there are two 2×3 tiles.

Christopher Jensen, in an unpublished paper, showed that if *two* cells are taken from a corner of any board, as shown in Figure 9, the board obviously cannot be tiled with trominoes. However, if none of these five cases is allowed, a $3m - 1$ by $3n + 1$ board, with any two cells missing can be tiled if an only if $n = 1$ or m and n are each equal to or greater than 3.

A final word

Kate Jones, who founded and runs Kadon Enterprises, a firm that makes and sells handsome mechanical puzzles, games, and other recreational math items, has on the market a game called Vee-21 [**5**]. The Vee is for V-trominoes, and 21 for the 21 tromino tiles in the set. The trominoes are brightly colored, and there is an order-8 board on

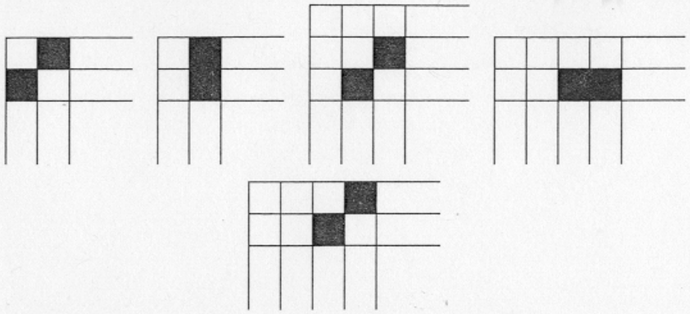

Figure 9. Impossible tiling patterns when *two* cells are missing at a corner.

which to place them. The basic task is to put a monomino (order-1 tile) at any spot on the board, then cover the remaining 63 cells with the trominoes, thus solving an order-8 board. A 40-page brochure comes with the set. It contains a short article on "The Deficient Checkerboard" by Norton Starr, and pictures of rectangular fields that offer other challenges.

Our final tiling (see Figure 10) is a beautiful, symmetric tiling of the standard chessboard.

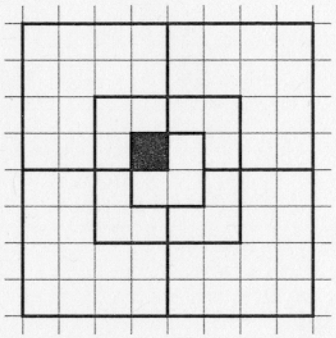

Figure 10. An order-8 tiling with no 2×3 tiles and 5 rep-tiles.

References

1. I. P. Chu and R. Johnsonbaugh, Tiling deficient boards with trominoes, *Math. Mag.* **59** (1986) 34–40.
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3. S. W. Golomb, Checker boards and polyominoes, *Amer. Math. Monthly* **61** (1954) 675–682.
4. ———, *Polyominoes*, Scribner, New York, 1965.
5. K. Jones, Vee-21; available at `http://www.gamepuzzles.com/polycub2.htm#V21`.
6. R. Nelsen, *Proofs Without Words II: More Exercises in Visual Thinking*, Mathematical Association of America, Washington, 2000.

Polyomino Problems to Confuse Computers

Stewart Coffin

After a brief career in electronics and manufacturing, **Stewart Coffin** turned his lifelong interest in mechanical puzzles from a hobby into a business, designing unusual geometrical puzzles and crafting them in fine woods. Now retired, he has recently written about them in his book *Geometric Puzzle Design.* He lives in Andover, Massachusetts.

For well over a century, puzzle pieces consisting of squares joined together all different ways have provided vexation for would be solvers by perversely declining to make room for each other inside a square or rectangular tray. In recent years, these lovable little pieces have played an increasing role in recreational mathematics. Solomon Golomb, in his 1965 book on the subject, referred to such shapes as polyominoes, and so by that name they are now commonly known.

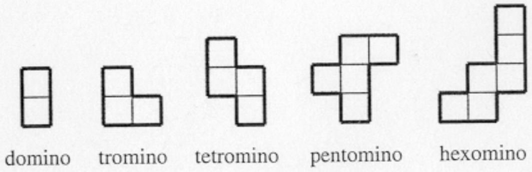

Figure 1. Some polyominoes.

Early on, one popular recreation was counting the possible polyominoes for increasing numbers of squares. Another was discovering which sets of pieces would assemble into various shaped solutions and in how many different ways, or proving that no such solution was possible. Now that so much of this can be done virtually instantly by computer, perhaps a better term for this would be *electronic* rather than *mathematical* recreations. A search for "polyominoes" on the Internet will reveal many examples of solutions involving very large numbers of pieces, most of which one may assume were arrived at by computer. But for practical sets of puzzle pieces, simpler is usually better. Consider, for example, the popular set of twelve pentominoes.

Into how many different rectangular trays can these pieces be packed solid, and in how many different ways? Since these are typically in the form of physical puzzle pieces, one assumes they can be rotated and turned over. They can form four different rectangles: 3×20, 4×15, 5×12 and 6×10. The complete analysis of possible solutions to these four shapes dates from around 1960 and marks one of the earliest uses of computers for solving problems of this sort. Even with 2339 solutions to the 6×10, finding just one of them can be frustrating for the novice. But one becomes much more efficient at this with studious practice, which can be an enjoyable recreation in itself. Just don't expect to compete with the computer for speed. One program now in use, Puzzlesolver3D, finds all 2339 solutions at the average rate of one every 85 milliseconds, even on the author's ancient computer.