Exam 1 Real Analysis 1 10/8/2010

Each problem is worth 10 points. Show adequate justification for full credit. Don't panic.

1. State the definition of the limit of a function $f(x)$ as x approaches $+\infty$.

2. a) State the definition of an accumulation point.

   b) Give an example of a subset of $\mathbb{R}$ with infinitely many elements but no accumulation points.

3. Give an example of a sequence which is bounded but does not converge.

4. State the Cauchy Convergence Criterion.

5. Prove that if a sequence has a limit, then that limit is unique.

6. Suppose that f and g are functions with both having domain $D \subseteq \mathbb{R}$. Prove that if $\lim_{x \to a} f(x) = A$ and $\lim_{x \to a} g(x) = B$ then $\lim_{x \to a} (f - g)(x) = A - B$.

7. State and prove the Bolzano-Weierstrass Theorem for Sets.

8. Using some or all of the axioms:

(A1) (*Closure*) $a + b$, $a \cdot b \in \mathbb{R}$ for any a, $b \in \mathbb{R}$. Also, if a, b, c, $d \in \mathbb{R}$ with $a = b$ and $c = d$, then $a + c = b + d$ and $a \cdot c = b \cdot d$.

(A2) (*Commutative*) $a + b = b + a$ and $a \cdot b = b \cdot a$ for any a, $b \in \mathbb{R}$.

(A3) (*Associative*) $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for any a, b, c, $\in \mathbb{R}$.

(A4) (*Additive identity*) There exists a zero element in $\mathbb{R}$, denoted by 0, such that $a + 0 = a$ for any $a \in \mathbb{R}$.

(A5) (*Additive inverse*) For each $a \in \mathbb{R}$, there exists an element $-a$ in $\mathbb{R}$, such that $a + (-a) = 0$.

(A6) (*Multiplicative identity*) There exists an element in $\mathbb{R}$, which we denote by 1, such that $a \cdot 1 = a$ for any $a \in \mathbb{R}$.

(A7) (*Multiplicative inverse*) For each $a \in \mathbb{R}$ with $a \neq 0$, there exists an element in $\mathbb{R}$ denoted by $\frac{1}{a}$ or a^{-1}, such that $a \cdot a^{-1} = 1$.

(A8) (*Distributive*) $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ for any a, b, $c \in \mathbb{R}$.

(A9) (*Trichotomy*) For a, $b \in \mathbb{R}$, exactly one of the following is true: $a = b$, $a < b$, or $a > b$.

(A10) (*Transitive*) For a, $b \in \mathbb{R}$, if $a < b$ and $b < c$, then $a < c$.

(A11) For a, b, $c \in \mathbb{R}$, if $a < b$, then $a + c < b + c$.

(A12) For a, b, $c \in \mathbb{R}$, if $a < b$ and $c > 0$, then $ac < bc$.

Prove that if a, $b \in \mathbb{R}$, and $ab = 0$, then $a = 0$ or $b = 0$ (or both). Be explicit about which axioms you use.

9. Show that if a is an accumulation point of a set S, then for any $\varepsilon > 0$, $(a - \varepsilon, a + \varepsilon)$ contains infinitely many points of S.

10. a) Show that $\lim_{x \to a} x = a$

b) Show that for any $n \in \mathbb{Z}_+$, $\lim_{x \to a} x^n = a^n$.