Exam 1 Real Analysis 1 10/8/2010

Each problem is worth 10 points. Show adequate justification for full credit. Don't panic.

1. State the definition of the limit of a function $f(x)$ as x approaches $+\infty$.

$$\lim_{x\to\infty} f(x) = L$$

Let $f(x)$ be a function with domain $D \in \mathbb{R}$, where D is not bounded above. We say the limit of $f(x)$ is L as x approaches $+\infty$ iff $\forall \varepsilon > 0 \ \exists M \in \mathbb{R} \ni x \geq M$ and $x \in D \Rightarrow |f(x) - L| < \varepsilon$.

Nice!

2. a) State the definition of an accumulation point.

We call a an accumulation point of a set S iff $\forall \varepsilon > 0 \ \exists_s \in S \ni 0 < |a - s| < \varepsilon$

Great

b) Give an example of a subset of $\mathbb{R}$ with infinitely many elements but no accumulation points.

The integers. $\mathbb{Z}$

Nice!

3. Give an example of a sequence which is bounded but does not converge.

$\{a_n\} = \{(-1)^n\}$

Great.

4. State the Cauchy Convergence Criterion.

In $\mathbb{R}$ a sequence is Cauchy iff convergent

Exactly.

5. Prove that if a sequence has a limit, then that limit is unique.

let ~~for~~ sequence $\{a_n\}$; $\lim_{n\to\infty} a_n = A$ & $\lim_{n\to\infty} a_n = B$

To prove $A = B$.

proof: Let $\varepsilon > 0$ be given,

$\lim_{n\to\infty} a_n = A \Rightarrow$ For each $\varepsilon > 0$, $\exists\, n_1 \in N \ni |a_n - A| < \frac{\varepsilon}{2}$ $\forall n \geq n_1$

$\lim_{n\to\infty} a_n = B \Rightarrow$ For each $\varepsilon > 0$, $\exists\, n_2 \in N \ni |a_n - B| < \frac{\varepsilon}{2}$ $\forall n \geq n_2$

let $n^* = \max\{n_1, n_2\}$

Very nice job!

Then $\forall\ n \geq n^*$

$$|A - B| = |A - a_n + a_n - B| \leq |A - a_n| + |a_n - B|$$
$$\leq |a_n - A| + |a_n - B|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$
$$< \varepsilon$$

Now,

Since $|A - B| < \varepsilon$ $\forall\ n \geq n^*$ $A = B$

6. Suppose that f and g are functions with both having domain $D \subseteq \mathbb{R}$. Prove that if $\lim_{x \to a} f(x) = A$ and $\lim_{x \to a} g(x) = B$ then $\lim_{x \to a}(f - g)(x) = A - B$.

Let $f, g : D \to \mathbb{R}$ with $D \subseteq \mathbb{R}$, $\lim_{x \to a} f(x) = A$, and $\lim_{x \to a} g(x) = B$. $f - g$ has the same domain as f and g, so since $\lim_{x \to a} f(x)$ exists, a is an accumulation point of D, which is the domain of $f - g$. Let $\varepsilon > 0$. Because $\lim_{x \to a} f(x) = A$, there exists a $\delta_1 > 0$ such that if $0 < |x - a| < \delta_1$ and $x \in D$, then $|f(x) - A| < \frac{\varepsilon}{2}$. Similarly, since $\lim_{x \to a} g(x) = B$, there exists a $\delta_2 > 0$ such that if $0 < |x - a| < \delta_2$ and $x \in D$ then $|g(x) - B| < \frac{\varepsilon}{2}$. Take $\delta = \min\{\delta_1, \delta_2\}$. Then if $0 < |x - a| < \delta$ and $x \in D$ we have $|f(x) - A| < \frac{\varepsilon}{2}$ and $|g(x) - B| < \frac{\varepsilon}{2}$. Hence $|(f-g)(x) - (A - B)| = |(f(x) - g(x)) - (A - B)| = |(f(x) - A) - (g(x) - B)| \leq |f(x) - A| + |g(x) - B| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, the first inequality in the previous expression being justified by the triangle inequality. Therefore by the definition of the limit of a function at a real number, $\lim_{x \to a}(f - g)(x) = A - B$ as required.

Excellent + !

7. State and prove the Bolzano-Weierstrass Theorem for Sets.

any infinite and bounded set has at least one accumulation point.

Proof: Well, let $\varepsilon > 0$ be given. Then the interval $[a_1, b_1]$ contains our set. Take $c_1 = \frac{a_1+b_1}{2}$.

Then either $[a_1, c_1]$ or $[c_1, b_1]$ contains infinitely many points of our set. Call that interval $[a_2, b_2]$ and repeat. Then we have $a_1 \leq a_2 \leq a_3 \leq \dots \leq a_n < b_n \leq \dots \leq b_3 \leq b_2 \leq b_1$

Thus $\{a_n\}$ is increasing and bounded. Therefore by the Monotone Convergence Thm $\{a_n\}$ converges to A. Similarily, $\{b_n\}$ converges to B. Then $A = B$ by the Squeeze Thm. Plus $|[a_n, b_n]| = \frac{1}{2^{n-1}} |[a_1, b_1]|$, which approaches zero, and thus A is within ε of other points of our set. Thus A is an accumulation point. $\square$

Nice.

8. Using some or all of the axioms:

(A1) (*Closure*) $a + b$, $a \cdot b \in \mathbb{R}$ for any $a, b \in \mathbb{R}$. Also, if $a, b, c, d \in \mathbb{R}$ with $a = b$ and $c = d$, then $a + c = b + d$ and $a \cdot c = b \cdot d$.

(A2) (*Commutative*) $a + b = b + a$ and $a \cdot b = b \cdot a$ for any $a, b \in \mathbb{R}$.

(A3) (*Associative*) $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for any $a, b, c, \in \mathbb{R}$.

(A4) (*Additive identity*) There exists a zero element in $\mathbb{R}$, denoted by 0, such that $a + 0 = a$ for any $a \in \mathbb{R}$.

(A5) (*Additive inverse*) For each $a \in \mathbb{R}$, there exists an element $-a$ in $\mathbb{R}$, such that $a + (-a) = 0$.

(A6) (*Multiplicative identity*) There exists an element in $\mathbb{R}$, which we denote by 1, such that $a \cdot 1 = a$ for any $a \in \mathbb{R}$.

(A7) (*Multiplicative inverse*) For each $a \in \mathbb{R}$ with $a \neq 0$, there exists an element in $\mathbb{R}$ denoted by $\frac{1}{a}$ or a^{-1}, such that $a \cdot a^{-1} = 1$.

(A8) (*Distributive*) $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ for any $a, b, c \in \mathbb{R}$.

(A9) (*Trichotomy*) For $a, b \in \mathbb{R}$, exactly one of the following is true: $a = b$, $a < b$, or $a > b$.

(A10) (*Transitive*) For $a, b \in \mathbb{R}$, if $a < b$ and $b < c$, then $a < c$.

(A11) For $a, b, c \in \mathbb{R}$, if $a < b$, then $a + c < b + c$.

(A12) For $a, b, c \in \mathbb{R}$, if $a < b$ and $c > 0$, then $ac < bc$.

Prove that if $a, b \in \mathbb{R}$, and $ab = 0$, then $a = 0$ or $b = 0$ (or both). Be explicit about which axioms you use.

Lemma: For any $c \in \mathbb{R}$, $c \cdot 0 = 0$.

Proof:
$$c \cdot 0 = c \cdot (c^{-1} + -c^{-1}) \quad \text{by A5}$$
$$= c \cdot c^{-1} + -c \cdot c^{-1} \quad \text{by A3 and an exercise}$$
$$= 1 + -1 \quad \text{by A7}$$
$$= 0 \quad \text{by A5}$$

Now for the main proof: Well, s'pose $ab = 0$ but $a \neq 0$.
Then by A7 we have an a^{-1}, and multiplying $ab = 0$ by it gives
$$a^{-1}(ab) = a^{-1} \cdot 0$$, which by A3 and the Lemma is
$$(a^{-1} \cdot a) \cdot b = 0$$, which by A7 is
$$1 \cdot b = 0$$, which by A6 is
$$b = 0.$$
Thus if $a \neq 0$, we have $b = 0$, so our result holds. □

9. Show that if a is an accumulation point of a set S, then for any $\varepsilon > 0$, $(a - \varepsilon, a + \varepsilon)$ contains infinitely many points of S.

Well, s'pose a is an accumulation point of S but that for some $\varepsilon > 0$, $(a-\varepsilon, a+\varepsilon)$ contains only finitely many points of S. Then the set of distances from a to each of these points is finite, so take the minimum value in this set, and call it ε^*. Then $(a-\varepsilon^*, a+\varepsilon^*)$ is a neighborhood of a containing no points from S, a contradiction. $\square$

10. a) Show that $\lim_{x \to a} x = a$

Let $\varepsilon > 0$ be given and $\delta = \min\{1, \varepsilon\}$ Scratch work

want: $|x-a| < \varepsilon$

so $0 < |x-a| < \delta = \varepsilon$

$\delta = \varepsilon$

$|x-a| < \varepsilon$

so $\lim_{x \to a} x = a$ as desired (by definition)

b) Show that for any $n \in \mathbb{Z}_+$, $\lim_{x \to a} x^n = a^n$. Induction!

so our base case $n=1$ is done above an shown true

assume our hypothesis: $\overset{n=k}{\lim_{x \to a} x^k = a^k}$

prove $k+1$ case: we want: $\lim_{x \to a} x^{k+1} = a^{k+1}$ so $x^{k+1} = x^k \cdot x$

let $f(x) = x^k$ and $g(x) = x$, so we have

$\lim_{x \to a} f(x) = a^k = $ "A" $\lim_{x \to a} g(x) = a = $ "A" by a theorem

proven in class if $\lim_{x \to a} f(x) = A$ and $\lim_{x \to a} g(x) = B$ then

$\lim_{x \to a} (f \cdot g)(x) = A \cdot B$ so for our case, $\lim_{x \to a} x^k \cdot x = a^k \cdot a$

and thus $\lim_{x \to a} x^{k+1} = a^{k+1}$ so since our $k+1$ case is proven

true our hypothesis is true, therefore $\lim_{x \to a} x^n = a^n$. □

Excellent!