Exam 2 Real Analysis 1 11/11/2010

Each problem is worth 10 points. Show adequate justification for full credit. Don't panic.

1. State the definition of a closed subset of $\mathbb{R}$.

A set $B \subseteq \mathbb{R}$ is closed iff all accumunation points of B are in the set B.

Good

2. State the (local) definition of continuity.

A function $f: D \to R$, $D \subseteq R$, is said to be continus at a, $a \in D$ iff $\forall \varepsilon > 0 \; \exists \delta > 0 \ni$

$|f(x) - f(a)| < \varepsilon$, provided $|x - a| < \delta$, $x \in D$.

Great.

3. a) State the definition of a compact set.

A set S is compact iff every open cover of S has a finite subcover.

b) State the Heine-Borel Theorem.

In $\mathbb{R}$, a set is compact iff it is closed and bounded.

Great!

4. State the Boundedness Theorem.

If $f: [a,b] \to \mathbb{R}$ is continuous, then f is bounded.

Good!

5. State and prove the Sum Rule for Derivatives.

Sum Rule:

f and g are differentiable functions at $x=a$

$\rightarrow (f+g)'(a) = f'(a) + g'(a)$

Proof:

By definition:
$$(f+g)'(a) = \lim_{x\to a} \frac{(f+g)(x) - (f+g)(a)}{x-a}$$

$$= \lim_{x\to a} \frac{f(x) + g(x) - (f(a) - g(a))}{x-a}$$

$$= \lim_{x\to a} \frac{(f(x) - f(a)) + (g(x) - g(a))}{x-a}$$

Excellent!

$$= \lim_{x\to a} \frac{f(x) - f(a)}{x-a} + \lim_{x\to a} \frac{g(x)-g(a)}{x-a}$$

(since f & g are both differentiable)

$$= f'(a) + g'(a)$$

(by definition)

$$\rightarrow \boxed{(f+g)'(a) = f'(a) + g'(a)}$$

6. State and prove Brouwer's Fixed Point Theorem.

Statement : If $f:[a,b] \to [a,b]$ is continous, then $\exists$ atleast one fixed point, i.e, $\exists$ atleast one real number $c \in [a,b]$ such that $f(c)=c$.

proof :

Case 1: If $f(a)=a$ and/or $f(b)=b$, proved.

Case 2: If $f(a) \neq a$ & $f(b) \neq b$

$\Rightarrow f(a) > a$ & $f(b) < b$ [as image of f lies on $[a,b]$]

Now,

let $g(x) = f(x) - x \quad \forall x \in [a,b]$

then, $g(x)$ is continous at $[a,b]$, differentiable at (a,b)

& $g(a) = f(a) - a > 0$ [as $f(a) > a$]

& $g(b) = f(b) - b < 0$ [as $f(b) < b$]

Then, by Bolzano Intermediate Value theorm,

[as 0 is between $g(a)$ & $g(b)$]

$\exists c \in [a,b] \ni g(c) = 0$

$\Rightarrow g(c) = f(c) - c = 0$

or, $f(c) = c$

Excellent

$\square$

7. State and prove the Extreme Value Theorem.

Let $f:[a,b] \to \mathbb{R}$ be continuous. Then f attains its max and min values on $[a,b]$ i.e. $\exists c \in [a,b]$ such that $f(c)$ is a max and a $d \in [a,b]$ for which $f(d)$ is a min.

Proof: (Max Case) Well By Boundedness Thm, we know f is bounded above and by Completeness, f has a least upper bound; Let's call it M. Now suppose there is not a $c \in [a,b]$ for which $f(c) \geq f(x)$ $\forall x \in [a,b]$. Then define $g(x) = \frac{1}{M-f(x)}$. Note that $g(x)$ is continuous on $[a,b]$ and thus has an upper bound, Let's call it K. Then $\forall x \in [a,b]$, $\frac{1}{M-f(x)} \leq K$ or $\frac{1}{K} \leq M-f(x)$ or $f(x) \leq M - \frac{1}{K}$ $\forall x \in [a,b]$. But this contradicts the fact that M was a least upper bound. Thus there must exist a $c \in [a,b]$ such that $f(c) \geq f(x)$ $\forall x \in [a,b]$. $\square$

Well done!

8. State and prove Rolle's Theorem.

Rolle's Theorem: If $f:[a,b] \to \mathbb{R}$ and f is continuous on $[a,b]$ and differentiable on (a,b) and $f(a)=f(b)$, then $\exists c \in (a,b) \ni f'(c)=0$.

Trivial case: f is a constant function. then any $c \in (a,b)$ will suffice to have $f'(c)=0$.

Max Case: S'pose $f(x)$ was not constant so $\exists x \in [a,b] \ni f(x) > f(a)$
Well, by the extreme value theorem, since f is continuous on $[a,b]$ f will attain an absolute max at some $c \in [a,b] \ni f(c) \geq f(x)$ $\forall x \in [a,b]$
Note $c \neq a$ or b as $\exists x \in (a,b) \ni f(x) > f(a) = f(b)$ so $f(c) > f(a)$
So by Fermat's theorem, since c is an extremum and $c \in (a,b)$ which f is differentiable on (a,b) Then $f'(c)=0$.

Min Case: S'pose $f(x)$ wasn't constant and $\exists x \in [a,b] \ni f(x) < f(a)$.
Similar logic will follow from above by the Extreme Value Theorem to show $\exists c \in (a,b)$ that is an absolute min of f on $[a,b]$ and Then Fermat's will prove $f'(c)=0$ as c was an extremum in (a,b)

So the proof is complete. $\exists c \in (a,b) \ni f'(c)=0$

Nice!

9. State and prove Fermat's Theorem.

Theorem: If $f: D \to \mathbb{R}$ has a local extremum at $c \in (a,b) \subseteq D$ and $f'(c)$ exists, then $f'(c)=0$.

Proof: Well, let's do the case where f has a local max at c, so there exists $\delta>0 \ni |x-c|<\delta \Rightarrow f(x) \leq f(c)$. Thus for $h \in (\delta,-\delta)$, $c+h \in D \Rightarrow f(c+h)-f(c) \leq 0$. Therefore

$$\frac{f(c+h)-f(c)}{h} \leq 0 \quad \text{if } h>0$$

and

$$\frac{f(c+h)-f(c)}{h} \geq 0 \quad \text{if } h<0$$

Hence $\displaystyle\lim_{h\to 0^+} \frac{f(c+h)-f(c)}{h} \leq 0$

and $\displaystyle\lim_{h\to 0^-} \frac{f(c+h)-f(c)}{h} \geq 0$

But since f is differentiable at c these must be equal, so $f'(c)=0$ as desired. □

10. The Racetrack Principle: Let f and g be differentiable functions from $[a, b]$ to $\mathbb{R}$, and suppose $f(a) = g(a)$. Show that if $f'(x) \geq g'(x)$ for all $x \in [a, b]$, then $f(x) \geq g(x)$ for all $x \in [a, b]$.

Well, sipose it's not the case that $f(x) \geq g(x)\ \forall x \in [a, b]$, so then we'd have some $c \in [a, b]$ for which $f(c) < g(c)$. Note $c \neq a$, since $f(a) \not< g(a)$, so actually $c \in (a, b]$. Let's make a new function h, where $h(x) = f(x) - g(x)$, so $h(c) = f(c) - g(c) < 0$, and $h(a) = f(a) - g(a) = 0$. Also $c - a > 0$, since $c \in (a, b]$, so

$$\frac{h(c) - h(a)}{c - a} < 0 \qquad *$$

But then note that h is also differentiable, and therefore continuous, on $[a, c]$, meeting the requirements for the Mean Value Theorem, so $\exists d \in (a, c) \ni h'(d) = \frac{h(c) - h(a)}{c - a}$. But then $h'(d) < 0$ by $*$, so $f'(d) - g'(d) < 0$, or $f'(d) < g'(d)$, contradicting one of our hypotheses. Then such a c must not exist after all, so $f(x) \geq g(x)$ $\forall x \in [a, b]$ as desired. $\square$