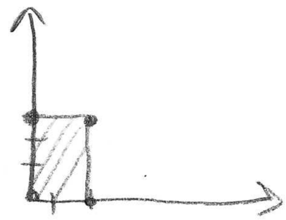
Exam 2 Calc 3 10/28/2011

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Set up an iterated integral for the volume under $f(x, y) = 3x^2 + 5x - 2y + 10$ and above the rectangle with vertices (2,0), (2,3), (0,3), and (0,0).



$$\int_0^2 \int_0^3 3x^2+5x-2y+10 \, dy\, dx$$

Good

2. The table below shows data from a population survey done on brown barbaloots in a region $R = [0,8] \times [0,4]$, given in barbaloots per square furlong. Estimate the total brown barbaloot population in this region using the Midpoint Rule with $m = n = 2$.

x \ y	0	1	2	3	4
0	12	14	15	14	7
2	13	16	21	20	11
4	14	17	23	19	12
6	11	13	21	18	9
8	8	9	10	7	5

$f(x,y)\Delta A$ $\quad$ $\Delta A = 8$

$f(1,2)8 + f(3,2)8$

$+ f(1,6)8 + f(3,6)8$

$(16\cdot 8) + (20\cdot 8) + (13\cdot 8) + (18\cdot 8)$

536

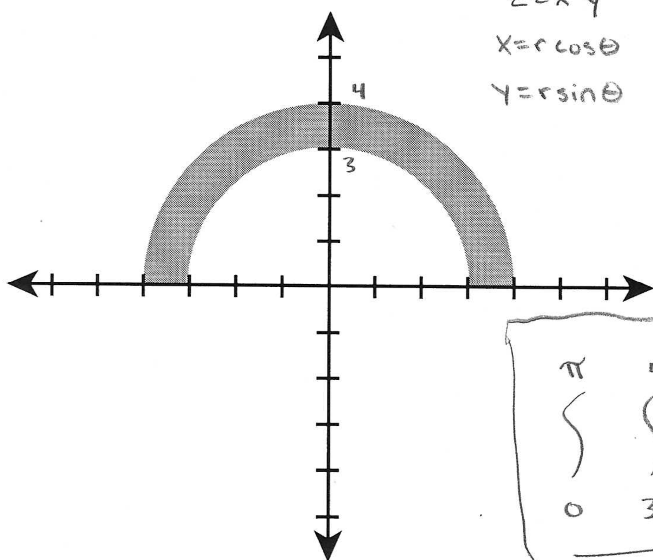
Great

3. Set up an iterated integral for the volume below $z = x^2y$, above the region shown below. Set up in terms of a single coordinate system, i.e., if you use cylindrical your integral should involve no x or y, etc.

$z = x^2 y$ $\quad z = (r\cos\theta)^2\, r\sin\theta$

$x = r\cos\theta$ $\quad z = r^3\cos^2\theta\,\sin\theta$

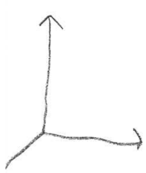
$y = r\sin\theta$

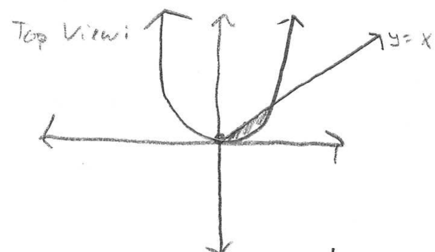


$$\int_0^{\pi}\int_3^{4}\int_0^{r^3\cos^2\theta\sin\theta} r\,dz\,dr\,d\theta$$

Good!

4. Set up an iterated integral for the volume of the region under the plane $x + 2y - z = 0$ and above the region bounded by $y = x$ and $y = x^4$.





$z = x + 2y$

$x = x^4$

$0 = x^4 - x$

$0 = x(x^3 - 1)$

$x = 0, 1$

$$\int_0^1\int_{x^4}^{x}\int_0^{x+2y} 1\,dz\,dy\,dx$$

Great!

5. Show that the Jacobian for converting from rectangular to polar coordinates is r.

$x = r\cos\theta \quad , \quad y = r\sin\theta$

$$\text{Jacobian} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix}$$

Well done

$$= r\cos^2\theta + r\sin^2\theta = r(\sin^2\theta + \cos^2\theta)$$

$$= r(1) = r$$ So the Jacobian is r

6. Set up an iterated integral for $\iiint_E xyz\, dV$, where E lies between spheres with radius 2 and 4, both centered at the origin, and above the cone $z = \sqrt{x^2 + y^2}$. Set up in terms of a single coordinate system, i.e., if you use cylindrical your integral should involve no x or y, etc.

Spherical $\quad z = \sqrt{x^2 + y^2}$

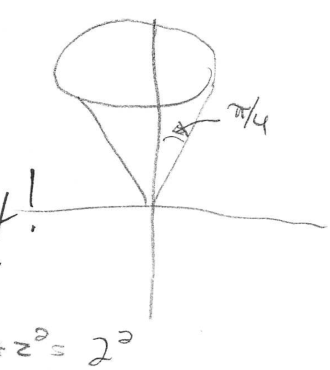
$$\iiint_E xyz\, dV$$

$0 \le \theta \le 2\pi$

$0 \le \phi \le \pi/4$

$2 \le \rho \le 4$

$$\int_0^{\pi/4}\int_0^{2\pi}\int_2^4 xyz\, dV$$



Excellent!

$xyz = \rho\sin\phi\cos\theta\,\rho\sin\phi\sin\theta\,\rho\cos\phi$

$(\rho^3\sin^2\phi\cos\phi\cos\theta\sin\theta)\,dV$

$x^2 + y^2 + z^2 = 2^2$

$x^2 + y^2 + z^2 = 4^2$

$\rho^2 = x^2 + y^2 + z^2$

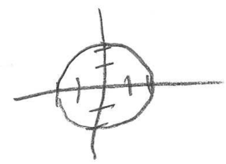
$dV = \rho^2\sin\phi\, d\rho\, d\theta\, d\phi$

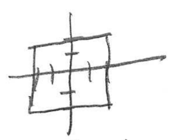
$$\int_0^{\pi/4}\int_0^{2\pi}\int_2^4 \rho^5\sin^3\phi\cos\phi\cos\theta\sin\theta\, d\rho\, d\theta\, d\phi$$

7. Bunny is a calculus student at Enormous State University, and she's having some trouble. Bunny says "Ohmygod, this Calc 3 stuff is soooo confusing! Like, we just started doing these double integral thingies, right? And the professor was, like, making fun of this girl for asking why if you go from –2 to 2 and –2 to 2, you don't get the volume of a paraboloid that's, like, $z = 4 - x^2 - y^2$. I mean, if you do less than that, you're leaving out part of the paraboloid, right?"

Explain clearly to Bunny what goes wrong when trying to find the volume of her paraboloid that way.

Well, if you are taking the volume of a paraboloid, or anything else for that matter, the first thing you want to do is look at the top view.

The top view of the paraboloid $z = 4 - x^2 - y^2$ is  a circle with radius 2.

If you draw out a shape from -2 to 2 and -2 to 2, you get  instead. The integral will go all the way out to 2 and -2 if you just take the limits like that. You have to add in the equation of a circle, so the y limits of integration should by $\sqrt{4-x^2}$ to $\sqrt{4-x^2}$. It is like if you want to find the rea of a circle you don't multiple 4×4, you have to take nto account the missing parts, or you'll ge the area of a rectangle instead.

Excellent!

8. Set up a triple integral for the volume of the portion of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ which lies above the plane $z = c/2$.

The ellipsoid and plane intersect where:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{\left(\frac{c}{2}\right)^2}{c^2}=1$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{1}{4}=1$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{3}{4}$$

← And from this the x-intercepts are:

$$\frac{x^2}{a^2}+\frac{(0)^2}{b^2}=\frac{3}{4}$$

$$x^2 = a^2\cdot\frac{3}{4}$$

$$x = \pm a\sqrt{\frac{3}{4}}$$

$$\frac{y^2}{b^2}=\frac{3}{4}-\frac{x^2}{a^2}$$

$$y = \pm b\sqrt{\frac{3}{4}-\frac{x^2}{a^2}}$$

So these give us limits for:

$$\text{Volume } \int_{x=-a\sqrt{\frac{3}{4}}}^{x=a\sqrt{\frac{3}{4}}}\int_{y=-b\sqrt{\frac{3}{4}-\frac{x^2}{b^2}}}^{y=b\sqrt{\frac{3}{4}-\frac{x^2}{b^2}}}\int_{z=\frac{c}{2}}^{z=c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} 1\,dz\,dy\,dx$$

9. Suppose that space aliens infuse the atmosphere with sillydust in such a way that the density of sillydust varies linearly with latitude from α ounces per cubic mile of atmosphere at the north pole to β ounces per cubic mile at the south pole, and drops off steadily to 0 at an altitude of 1 mile above the planet's surface. Modeling the Earth as a sphere with radius 4000 miles, set up an iterated integral for the amount of sillydust in Earth's atmosphere.

To get the pole-to-pole:

ϕ	δ
0	α
π	β

so $m = \dfrac{\beta-\alpha}{\pi - 0}$, and $\delta(\phi) = \alpha + \dfrac{\beta-\alpha}{\pi}\phi$ at the surface

To get the altitude:

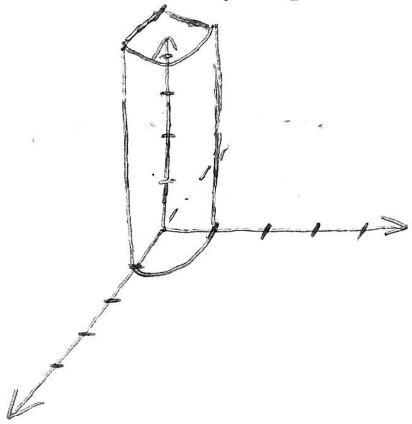
ρ	k
4000	1
4001	0

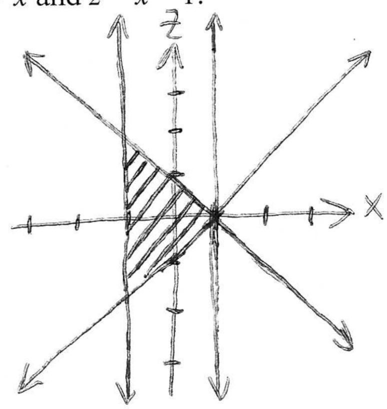
so $m = \dfrac{0-1}{4001-4000}$, and $k(\rho) = 4001-\rho$

Then $\delta(\phi,\rho) = \left(\alpha + \dfrac{\beta-\alpha}{\pi}\phi\right)(4001-\rho)$ is a density function that does what's described, so

$$\int_0^{2\pi}\int_0^{\pi}\int_{4000}^{4001}\left(\alpha + \frac{\beta-\alpha}{\pi}\phi\right)(4001-\rho)\rho^2\sin\phi \, d\rho \, d\phi \, d\theta$$

10. Set up integrals for the x coordinate of the centroid of the wedge of the cylinder $x^2 + y^2 = 1$ sliced off by the planes $z = 1 - x$ and $z = x - 1$.





$$\int_0^{2\pi}\int_0^1\int_{x-1}^{1-x} r\,dz\,dr\,d\theta$$

$$\bar{X} = \frac{\int_0^{2\pi}\int_0^1\int_{x-1}^{1-x} K(r\cos\theta)\,r\,dz\,dr\,d\theta}{\int_0^{2\pi}\int_0^1\int_{x-1}^{1-x} K\,r\,dz\,dr\,d\theta}$$

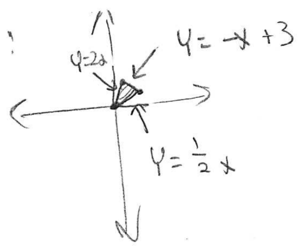
$$\bar{X} = \frac{\cancel{K}\int_0^{2\pi}\int_0^1\int_{x-1}^{1-x} r^2\cos\theta\,dz\,dr\,d\theta}{\cancel{K}\int_0^{2\pi}\int_0^1\int_{x-1}^{1-x} r\,dz\,dr\,d\theta}$$

Nice.

Your score on question 11 can replace your lowest score among questions 8-10, if desired.

11. Evaluate the integral $\iint_R (x-3y)\,dA$, where R is the triangular region with vertices (0,0), (2,1), and (1,2), using the transformation $x = 2u + v$, $y = u + 2v$.

Top View:



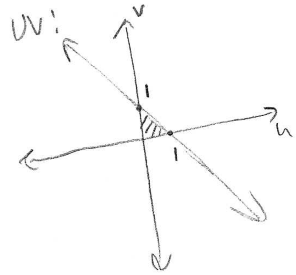
$u + 2v = 4u + 2v \Rightarrow u = 4u \Rightarrow u = 0$

$u + 2v = -2u - v + 3 \Rightarrow 3u + 3v = 3 \Rightarrow u + v = 1 \Rightarrow v = -u + 1$

$u + 2v = u + \frac{1}{2}v \quad 2v = \frac{1}{2}v \quad 1.5v = 0$

$v = 0$

Top View UV:



$$\begin{vmatrix} x_u = 2 & y_u = 1 \\ x_v = 1 & y_v = 2 \end{vmatrix} \quad 4 - 1 = 3$$

$u^2 - 2u + 1 \rightarrow \frac{u^3}{3} - u^2 + u$

$$3\int_0^1 \int_0^{-u+1} [-u - 5v]\,dv\,du = 3\int_0^1 \left[-uv - \frac{5v^2}{2}\right]_0^{-u+1} du$$

$$= 3\int_0^1 \left[u^2 - u - \frac{5(-u+1)^2}{2}\right] du = 3\left(\frac{1}{3} - \frac{1}{2}\right) - 3\left(\frac{5}{2}\right)\left(\frac{1}{3} - 1 + 1\right)$$

$= -3$

Excellent!