Exam 3 Calc 3 11/30/2012

Each problem is worth 10 points. For full credit provide complete justification for your answers.

x^5y^2 x^5y^2

1. Let $\mathbf{F}(x,y) = \langle 5x^4y^2, 2x^5 y\rangle$, and C be the line segment from (2,–1) to (0,2). Compute $\int_C \mathbf{F}\cdot d\mathbf{r}$.

$$\frac{\partial(5x^4y^2)}{\partial y} = \frac{\partial(2x^5y)}{\partial x}$$

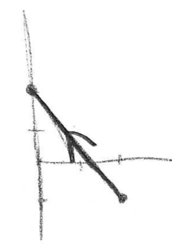
$$10x^4y = 10x^4y$$

Great

potential function is $\underline{x^5y^2}$

Through the Fun. Theorem:

$$x^5y^2 \Big|_{[2,-1]}^{[0,2]}$$



$$0^5 2^2 - 2^5(-1)^2 = \boxed{-32}$$

2. Let $\mathbf{F}(x, y, z) = \langle 5, 2xy, x - y + z\rangle$. Find div $\mathbf{F}$.

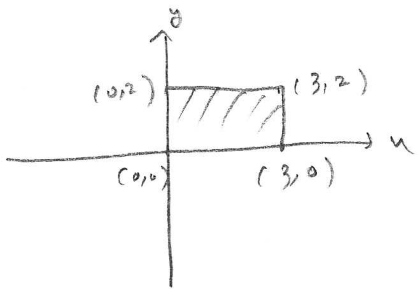
$$\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle 5, 2xy, x-y+z \rangle$$

$$\frac{\partial(5)}{\partial x} + \frac{\partial(2xy)}{\partial y} + \frac{\partial(x-y+z)}{\partial z}$$

$$0 + 2x + 1 = \boxed{2x+1}$$

Nice!

3. Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x,y) = \overset{P}{3xy}\mathbf{i} + \overset{Q}{6x^2}\mathbf{j}$ and C is the path consisting of four line segments joining the points (0,0), (3,0), (3,2), and (0,2) in that order.



Closed Path, So we can use Greens Theorem

$$\int_0^3\int_0^2 \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dy\, dx$$

$$\int_0^3\int_0^2 (12x - 3x)\, dy\, dx$$

$$\int_0^3\int_0^2 9x\, dy\, dx$$

Excellent!

$$\int_0^3 \left(9xy\Big|_0^2 \right) dx$$

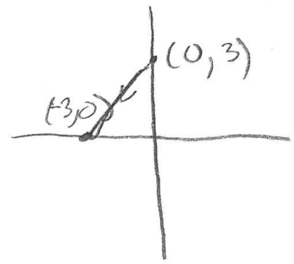
$$\int_0^3 18x\, dx$$

$$= \frac{\cancel{18}^{\,9}\, x^2}{\cancel{2}}\Big|_0^3$$

$$= 9x^2\Big|_0^3$$

$$= \underline{81}$$

4. Let $\mathbf{F}(x,y) = \langle xy + x^2, y^2\rangle$. Compute $\int_C \mathbf{F}\cdot d\mathbf{r}$ for C the second-quadrant portion of a circle with radius 3 centered at the origin, traversed counterclockwise.



$\int xy + x^2\,dx$

$\phi = x^2y + \frac{x^3}{3} + c(y)$

$\phi_y = x^2 + c_y(y)$ no Function

$r(t) = \langle 3\cos t, 3\sin t\rangle$ for $\frac{\pi}{2} \le t \le \pi$

$r'(t) = \langle -3\sin t, 3\cos t\rangle$

$F(r) = \langle 9\cos t\sin t + 9\cos^2 t, 9\sin^2 t\rangle$

$F\cdot dr = \langle 9\cos t\sin t + 9\cos^2 t, 9\sin^2 t\rangle \cdot \langle -3\sin t, 3\cos t\rangle$

$= \cancel{-27\cos t\sin^2 t} - 27\cos^2 t\sin t + \cancel{27\cos t\sin^2 t}$

$= -27\cos^2 t\sin t$

$$\int_{\frac{\pi}{2}}^{\pi} -27\cos^2 t\sin t\,dt$$

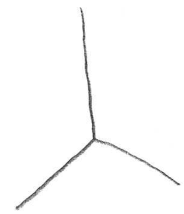
$9\cos^3 t\Big|_{\frac{\pi}{2}}^{\pi}$

well done

$= 9(-1)^3 - 0$

$= -9$

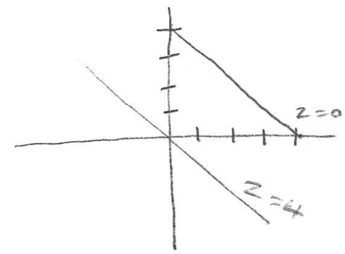
5. Let $\mathbf{F}(x, y, z) = \langle 0,0,-1 \rangle$, and S be the portion of $z = 4 - x - y$ in the first octant, oriented upward. Evaluate $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$.



long way

I $\quad r(u,v) = \langle u, v, 4-u-v \rangle$

II $\quad F(r(u,v)) = \langle 0, 0, -1 \rangle$



III $\quad r_u = \langle 1, 0, -1 \rangle$

$r_v = \langle 0, 1, -1 \rangle$

$$r_u \times r_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix}$$

$= \langle 1, 1, 1 \rangle$ ✓ oriented upward

IV $\quad \iint \langle 0, 0, -1 \rangle \cdot \langle 1, 1, 1 \rangle$

V $\quad \int_0^4 \int_0^{4-u} -1 \, dv\, du$

$-1v \Big|_0^{4-u}$

$\int_0^4 (-4+u)\, du$

$-4u + \frac{1}{2}u^2 \Big|_0^4$

$-16 + \frac{1}{2}(16)$

-8

Excellent!

6. Show that for any scalar function $f(x, y, z)$ with continuous second-order partial derivatives, $\text{curl}(\nabla f) = \mathbf{0}$. Make it clear why the requirement about continuity is important.

$$\nabla f = \langle f_x, f_y, f_z \rangle$$

$$\text{curl}(\nabla f) = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \times \langle f_x, f_y, f_z \rangle$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix}$$

$$= \langle f_{zy} - f_{yz}, -(f_{zx} - f_{xz}), f_{yx} - f_{xy} \rangle$$

by Clairot's theorem for any scalar function w/ continous second-order partial derivatives

$$f_{zy} = f_{yz}$$
$$f_{zx} = f_{xz}$$
$$f_{yx} = f_{xy}$$

Nice!

$$\therefore \langle f_{zy} - f_{yz}, -(f_{zx} - f_{xz}), f_{yx} - f_{xy} \rangle = \langle 0, 0, 0 \rangle = \vec{0} \ \square$$

7. Biff is a student at Enormous State University, and he's having some trouble. Biff says "Man, I was doing okay with these line integrals when they just told us which way to do 'em, but then for the exam it turns out they don't tell you which section a problem is from, so you actually have to think for yourself. I hate that. So what I've got to figure out is how to know when I can use that fundamental theory thing for line integrals instead of doing them the long way. Got any advice?"

   Help Biff out by explaining as clearly as possible when he can use the Fundamental Theorem for Line Integrals on a problem.

I know, Biff. Thinking is just so hard! But really, the Fundamental theorem for Line Integrals makes the problem a lot easier, so there's less thinking (kind of)! You can use the Fundamental (or Fun. ☺) Theorem when the function of the line integral given has a potential function. A potential function is a function whose derivative with respect to x will give you the x component of the parametrized function given and the derivative with respect to y will give you the y component. In shorthand, $\vec{F} = \langle f_x, f_y \rangle$* for some function f. When that f exists, be very happy, because it's Fun! ☺

* This can also be extended to z and so on, where $\vec{F} = \langle f_x, f_y, f_z \rangle$, etc.

Excellent!