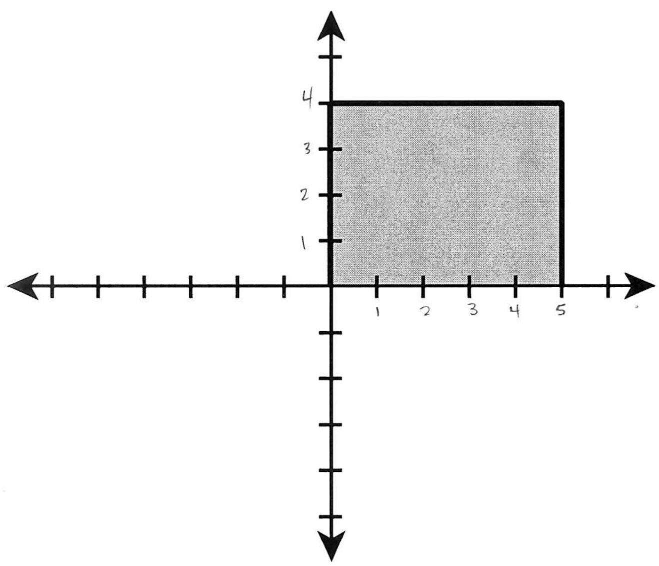
Exam 2b Calc 3 10/25/2013

Each problem is worth 10 points. For full credit provide complete justification for your answers. All integrals should be set up in terms of a single coordinate system, i.e., if you use cylindrical your integral should involve no x or y, etc.

1. Set up an iterated integral for the area of the rectangle shown (assuming tick marks are at unit intervals)

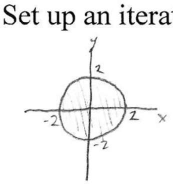
$$\boxed{\text{Area} = \int_{x=0}^{x=5}\int_{y=0}^{y=4} 1 \, dy\, dx}$$



Good

2. Set up an iterated integral for the volume under $z = 10 + x$ inside the circle $x^2 + y^2 = 4$.

Top View:



$$\boxed{\text{Volume} = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (10+x)\, dy\, dx}$$

$x^2+y^2=4$

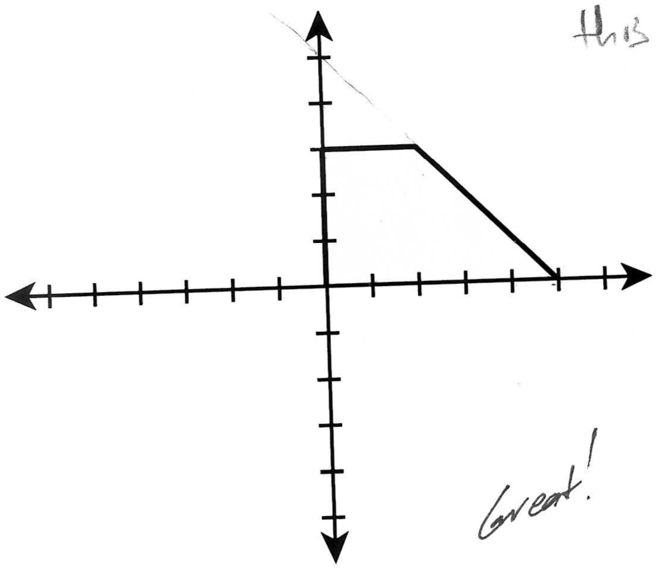
$y^2=4-x^2$

$y=\pm\sqrt{4-x^2}$

Great!

3. Set up an iterated integral for the volume below $z = 10 - 2x$, above the region shown below.

dx dy works best for this!



$y = -x + 5$

$x = 5 - y$

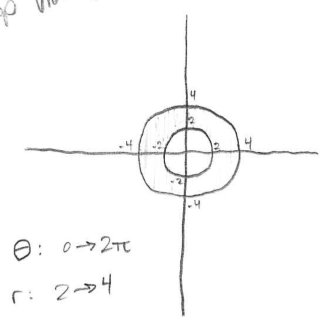
$$\int_{y=0}^{y=3} \int_{x=0}^{x=5-y} 10 - 2x \, dx \, dy$$

Great!

4. Set up an iterated integral for the volume of the region beneath the surface $z = xy + 10$ and above the annular region outside $x^2 + y^2 = 4$ but inside $x^2 + y^2 = 16$.

Polar coordinates will be simpler.

Top View:



$$\text{Volume: } \int_0^{2\pi} \int_2^4 (r^2 \sin\theta \cos\theta + 10) \, r \, dr \, d\theta$$

$\theta: 0 \to 2\pi$

$r: 2 \to 4$

$x^2 + y^2 = 16$ $\quad$ $x^2 + y^2 = 4$

$r^2 = 16$ $\quad$ $r^2 = 4$

$r = 4$ $\quad$ $r = 2$

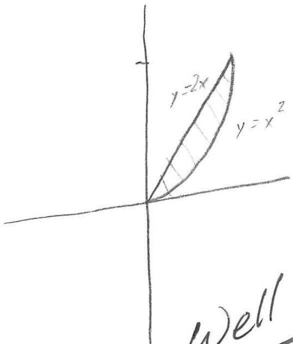
$z = xy + 10$

$z = r^2 \sin\theta \cos\theta + 10$

Excellent!

5. Reverse the order of integration in $\int_0^2\int_{x^2}^{2x} f(x,y)\,dy\,dx$.

Top view



$$2x = x^2 \qquad 2(2)=4$$
$$2 = x \qquad y=4$$

$$0 \leq y \leq 4$$

Well done!

$$y=2x \qquad y=x^2$$
$$x=y/2 \qquad x=\sqrt{y}$$

positive because we're only dealing with positive half.

$$\int_0^4\int_{y/2}^{\sqrt{y}} f(x,y)\,dx\,dy$$

6. Find the Jacobian for converting from rectangular to polar coordinates.

$$x=r\cos\theta \qquad y=r\sin\theta$$

$$J = \begin{vmatrix} \partial x/\partial r & \partial y/\partial r \\ \partial x/\partial\theta & \partial y/\partial\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

Excellent!

$$= r(\cos^2\theta + \sin^2\theta)$$
$$= r(1)$$
$$= r$$

Pythagorean Identity.

7. Bunny is a calculus student at Enormous State University, and she's having some trouble. Bunny says "Ohmygod, this Calc 3 stuff is soooo confusing! Like, our exam review sheet has a bunch of multiple guess questions, right? And one of them was, like, if you know a function has an average value of 2 for some region, then what's the max it could be someplace on that. How can I do that without a formula or anything?"

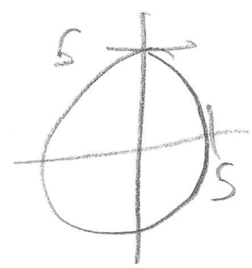
Explain clearly to Bunny what can be said about this, and why.

Well Bunny, it turns out that without any more information, the answer is unbounded. The only stipulation is that the average value is 2. That means it could get to 100 or 1000 or even higher somewhere so long as there is a different part of the function that balances it out. Therefore, the max could be any number greater than or equal to two.

Nice answer.

8. Set up iterated integrals for the z coordinate of the center of mass of the paraboloid below $z = 25 - x^2 - y^2$ but above $z = 0$.

Top view



Let's use cylindrical!

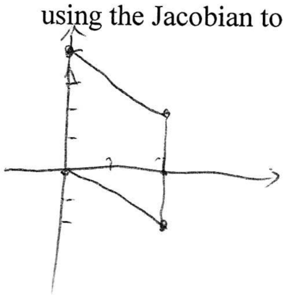
$$z = 25 - x^2 - y^2$$
$$z = 25 - r^2$$

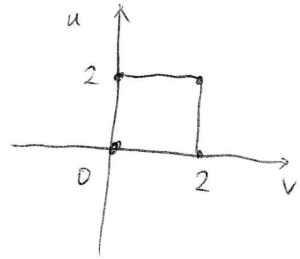
$$0 = 25 - r^2$$
$$0 \le r \le 5$$
$$0 \le \theta \le 2\pi$$

$$\bar{z} = \frac{\int_0^{2\pi}\int_0^{5}\int_0^{25-r^2} kzr \, dz\,dr\,d\theta}{\int_0^{2\pi}\int_0^{5}\int_0^{25-r^2} kr \, dz\,dr\,d\theta}$$

Yes.

9. Evaluate the integral $\iint_R xy\,dA$ on the region R with vertices (0,0), (2,2), (4,0), and (2,–2) by using the Jacobian to convert using the transformation $x = u + v$, $y = u - v$.





$0 = u+v$

$0 = u-v$

$v = u$

$2 = u+v \quad 2 = 2+v+v$

$2 = u-v \quad 0 = 2v$

$2+v = u \quad v = 0$

$u = 2$

$4 = u+v$

$0 = u-v$

$u = 2, \ v = 2$

$T: \ x = u+v \quad y = u-v$

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -1-1$$
$$= -2.$$

$$\iint_R xy\,dA = \int_0^2\int_0^2 (u+v)(u-v)\cancel{dv\,du}\cdot|J(u,v)|\cdot dv\,du$$

$$= \int_0^2\int_0^2 2u^2 - 2v^2\,dv\,du$$

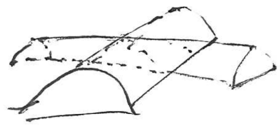
$$= \int_0^2 2u^2v - \frac{2v^3}{3}\Big|_0^2\,du$$

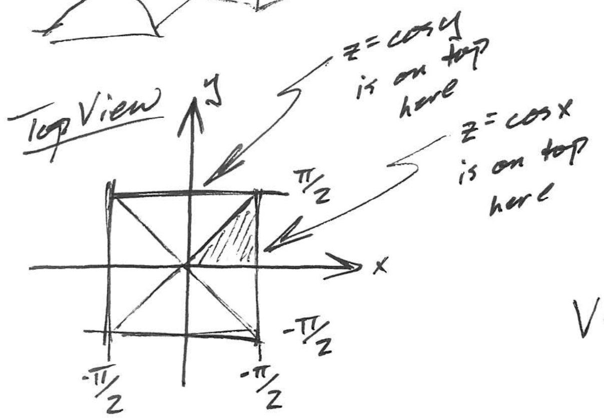
$$= \int_0^2 4u^2 - \frac{16}{3}\,du$$

$$= \frac{4}{3}u^3 - \frac{16}{3}u\Big|_0^2 = \frac{4}{3}\cdot 8 - \frac{16}{3}\cdot 2 = 0$$

Nice Job!

10. Set up an iterated integral (or integrals) for the volume of the region under **both** $z = \cos x$ and $z = \cos y$, but above the plane $z = 0$, and within 3 units of the origin.





$$V = 8\int_0^{\pi/2}\int_0^{x} \cos x \, dy\, dx$$

(There are other bumps farther out, but this is the only positive stuff within 3 units of the origin.)

The integral does the shaded part of the top view, and there are 8 pieces like that.