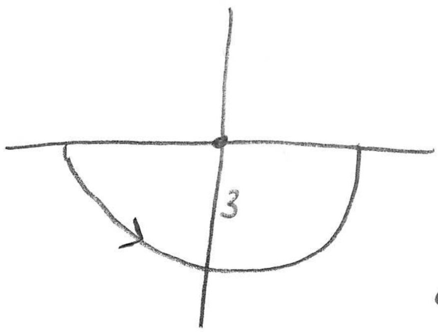
Exam 3b   Calculus 3   11/26/2013

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Give a parametrization and bounds for t for the bottom half of the circle with radius 3, centered at the origin and traversed counterclockwise.



$r = \langle 3\cos t, 3\sin t \rangle$

for $\pi \le t \le 2\pi$

Great

2. If $\mathbf{F}(x,y,x) = 3xy\,\mathbf{i} + 6yz\,\mathbf{j} + xz\,\mathbf{k}$, evaluate curl $\mathbf{F}$.

$$\text{Curl}(F) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3xy & 6yz & xz \end{vmatrix} = (0 - 6y)\vec{i} + (0 - z)\vec{j} + (0 - 3x)\vec{k}$$

Good!

$\langle -6y, -z, -3x \rangle$

f_{xy} f_{yx}
$x \neq 0$

3. Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x,y) = xy\,\mathbf{i} + 2y\,\mathbf{j}$ and C is the line segment from (0,0) to (3,4).

Long way

$r = \langle 3t, 4t \rangle$ for $0 \le t \le 1$

$F(r) = \langle 12t^2, 8t \rangle$

$\vec{r}\,' = \langle 3, 4 \rangle$

$\int F(r) \cdot \vec{r}\,' = \int_0^1 36t^2 + 32t\,dt$

Well done!

$12t^3 + 16t^2 \Big|_0^1 = 12 + 16 - (0 + 0)$

$= 28$

4. Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x,y) = 15x^4y\,\mathbf{i} + 3x^5\,\mathbf{j}$ and C is the line segment from (1,3) to (2,5).

$\vec{F} = \nabla f$, $f = 3x^5y$

by fundamental theorem for line integrals

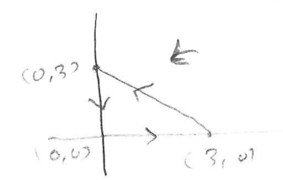
$\int_C \vec{F}\,d\vec{r} = 3x^5y \Big|_{(1,3)}^{(2,5)}$

Nice!

$= 3(2^5 \cdot 5 - 1^5 \cdot 3) = 3(157) = 471$

5. Let $\mathbf{F}(x,y) = 3xy\,\mathbf{i} + x^2\,\mathbf{j}$ and C be the line segment from (0,0) to (3,0), followed by the line segment from (3,0) to (0,3), then the line segment from (0,3) to the origin. Set up an integral (or integrals) involving only scalar quantities for $\int_C \mathbf{F}\cdot d\mathbf{r}$.

It's a closed curve, so using Green's theorem!



$$F(x,y) = \langle \underset{P}{3xy}, \underset{Q}{x^2} \rangle$$

$y-3=-1(x-0)$

$y=3-x$

$x+y=3$

$$\oint_C F\cdot dr = \iint_R (2x - 3x)\, dR$$

$$= \iint_R (-x)\, dR$$

Excellent!

$$\oint_C F\cdot dr = \int_0^3 \int_0^{3-x} -x \; dy\, dx$$

6. Show that for any vector field in $\mathbb{R}^3$ whose component functions have continuous second-order partial derivatives, div curl **F** = 0. Make it clear why the requirement about continuity is important.

Let $\vec{F} = \langle P, Q, R \rangle$

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix}$$

$$= \langle -Q_z + R_y,\ P_z - R_x,\ Q_x - P_y \rangle$$

$$\text{div}(\text{curl } \vec{F}) = (R_y - Q_z)_x + (P_z - R_x)_y + (Q_x - P_y)_z$$

$$= (R_{yx} - R_{xy}) + (P_{zy} - P_{yz}) + (Q_{xz} - Q_{zx})$$

Nice!

Since P, Q, R have continuous second-order partial derivatives, by Clairout's theorem:

$$\begin{cases} P_{zy} = P_{yz} \\ Q_{xz} = Q_{zx} \\ R_{yx} = R_{xy} \end{cases} \Rightarrow \begin{cases} P_{zy} - P_{yz} = 0 \\ Q_{xz} - Q_{zx} = 0 \\ R_{yx} - R_{xy} = 0 \end{cases}$$

$\Rightarrow \text{div}(\text{curl } \vec{F}) = 0$

7. Biff is a Calc 3 student at Enormous State University and he's having some trouble. Biff says "Dude, this stuff is killing me. I'm pretty good with it when you're dropping things into a formula, but now there's all these theorems and stuff. Our TA keeps saying things like how circulation is always zero in a conservative vector field, and I don't know what any of that means. Can you help me?"

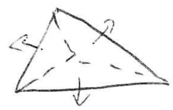
   Explain as clearly as possible to Biff what each of those terms means, and why his TA's conclusion is valid.

Dear Biff,

To understand what your TA said, you first have to understand the terms. Circulation means doing a line integral on a closed path, so if your curve is defined parametrically as $\vec{r}(x,y) = \vec{r}(t)$, then $\vec{r}(a) = \vec{r}(b)$ for $a \leq t \leq b$. A conservative vector field is a vector field that has a potential function. Because of this, doing a line integral on a conservative vector field is always path-independent, meaning we only evaluate the potential function at the end points. Now can you see why circulation is always zero in a conservative vector field?

Mathematically, $\int_a^b \vec{F}(r(t)) \cdot r'(t)\,dt = f(\vec{r}(b)) - f(\vec{r}(a))$, $= 0$ since $\vec{r}(a) = \vec{r}(b)$ as it is a circulation ($\vec{F} = \nabla f$). I hope that helps!

Excellent!



8. Suppose that **F** is a vector field whose divergence is zero everywhere and S is a tetrahedron consisting of the four sides S_1, S_2, S_3, and S_4. If you know that $\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = 2$, $\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = 2$, and $\iint_{S_3} \mathbf{F} \cdot d\mathbf{S} = 2$, then what can be said about $\iint_{S_4} \mathbf{F} \cdot d\mathbf{S}$? Explain your reasoning.

Let's consider $\iint_E \vec{F} d\vec{S}$, where $E = S_1 + S_2 + S_3 + S_4$ is the surface enclosing the tetrahedron.

By divergence theorem,

$$\iint_E \vec{F} d\vec{S} = \iiint_S \operatorname{div} \vec{F} \, dV$$

$$= \iiint_S 0 \, dV = 0$$

Since $\iint_E \vec{F} d\vec{S} =$

$$\iint_{S_1} \vec{F} d\vec{S} + \iint_{S_2} \vec{F} d\vec{S} + \iint_{S_3} \vec{F} d\vec{S} + \iint_{S_4} \vec{F} d\vec{S}$$

(all normal vectors point outward)

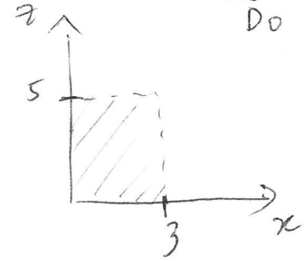
$$\Rightarrow \iint_{S_4} \vec{F} d\vec{S} = 0 - 2 - 2 - 2 = -6$$

Yes!

9. Suppose $\mathbf{F}(x, y, z) = P(x, y, z)\,\mathbf{i} + R(x, y, z)\,\mathbf{k}$, and let S be the rectangular portion of the plane $y = 2$ with vertices (0,2,0), (3,2,0), (3,2,5), and (0,2,5), oriented in the direction of the positive y-axis. Show why $\iint_S \mathbf{F} \cdot d\mathbf{S}$ must equal zero.

Do it the long way (with some shortcuts)

$\boxed{y=2}$



Since S lies in the plane $y = 2$

$\Rightarrow \hat{n} = \langle 0, 1, 0 \rangle$ (perpendicular to xz-plane).

$\hat{n}\, dS = \langle 0, 1, 0 \rangle \, dx\, dz$

$$\iint_S \vec{F}\, d\vec{S} = \iint_S \vec{F} \cdot \hat{n}\, dS$$

Wonderful Job

$$= \iint_S \langle P, 0, R \rangle \cdot \langle 0, 1, 0 \rangle \, dx\, dz$$

$$= \iint_S 0 \, dx\, dz$$

$= 0$ since integrand is 0

$yx - zx$ $\quad zy - xy \quad xz - yz$

10. Let $\mathbf{F}(x,y,z) = \langle y - z, z - x, x - y\rangle$. Let S be the portion of the sphere $x^2 + y^2 + z^2 = 25$ below $z = 4$, with outward orientation. Evaluate $\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}$.

(Webwork problem 2) sort of...

Good 'ol Stokes theorem

3, 0, 4

sphere radius 5 below $z=4$

$\int_0^{2\pi} 9 + 12\int_0^{2\pi}\sin t\,dt - 12\int_0^{2\pi}\cos t\,dt$

$18\pi + 12[\cos t]_0^{2\pi} + 12[\sin t]_0^{2\pi}$

$= \boxed{18\pi}$

$\iint_S \operatorname{curl} F\cdot dS = \int_C \vec{F}\cdot d\vec{r}$

$x^2 + y^2 + 16 = 25$

$x^2 + y^2 = 9$ circle radius 3

I. parametrize $0 \le t \le 2\pi$

$\langle 3\cos t, -3\sin t, 4\rangle$

II. $F(r(t))$

$\langle -3\sin t - 4, 4 - 3\cos t, 3\cos t + 3\sin t\rangle$

III. $r'(t) = \langle -3\sin t, -3\cos t, 0\rangle$

IV. $\int_C 9\sin^2 t + 12\sin t - 12\cos t + 9\cos^2 t + 0\,dt = \int_0^{2\pi} 9 + 12\sin t - 12\cos t\,dt$

$= \boxed{18\pi}$ Wonderful!