Exam 2 Real Analysis 1 11/7/2014

Each problem is worth 10 points. Show adequate justification for full credit. Don't panic.

1. State the local definition of continuity.

a function $f: D \to \mathbb{R}$ with $D \subseteq \mathbb{R}$ and $a \in D$ is continuous at a iff

$\forall \varepsilon > 0, \exists \delta > 0 \ni |x-a| < \delta$ and $x \in D \Rightarrow |f(x)-f(a)| < \varepsilon$

Great

2. a) State the definition of a relative maximum.

Let $f: D \to \mathbb{R}$, where $D \subseteq \mathbb{R}$. We say $a \in D$ is a relative maximum of f iff $\exists \delta > 0 \ni \forall x \in (a-\delta, a+\delta)$ we have $f(a) \geq f(x)$.

b) State Fermat's Theorem

If $f: D \to \mathbb{R}$ has a local extremum at $c \in (a,b) \subseteq D$ and $f'(c)$ exists, then $f'(c) = 0$.

3. a) Give an example of a function f that is differentiable at $x = a$ such that $f'(a)$ exists, with $f'(a) \neq 0$, but yet f attains a relative extremum at $x = a$.

$f: [0,1] \to \mathbb{R}$

$f(x) = x$

$a = 1$

b) Give an example of a function f that is continuous at $x = a$, not differentiable at $x = a$, but yet f attains a relative extremum at $x = a$.

$f: [-1,1] \to \mathbb{R}$

$f(x) = |x|$ Great

$a = 0$

local min at 0

4. a) State the definition of a compact set.

A set X is compact iff every open cover of X has a finite subcover.

b) State the Heine-Borel Theorem.

Every set $X \subseteq \mathbb{R}^n$ is compact iff X is closed and bounded

c) Give an example of a set with an open cover that has no finite subcover.

Let $X = (0,1)$, and open cover of X be the family of sets

$\mathcal{U} = \{ (\frac{1}{n}, 1) \mid n \in \mathbb{N} \}$

Then $\mathcal{U}$ ~~k~~ is an open cover of X with no finite subcover.

Excellent!

5. State and prove the Difference Rule for Derivatives.

Suppose $f, g : D \to \mathbb{R}$, $D \subseteq \mathbb{R}$ are two functions that are differentiable on D. Then $f-g : D \to \mathbb{R}$ is differentiable on D, and

$$(f-g)'(a) = f'(a) - g'(a)$$

for $\forall a \in D$.

Proof

Let $a \in D$, ~~and~~ then

$$\lim_{x \to a} \frac{(f(x) - g(x)) - (f(a) - g(a))}{x-a}$$

$$= \lim_{x \to a} \frac{f(x) - f(a)}{x-a} - \lim_{x \to a} \frac{g(x) - g(a)}{x-a} \quad \text{(provided these limits exist)}$$

$$= f'(a) - g'(a)$$

Hence $\lim_{x \to a} \frac{(f-g)(x) - (f-g)(a)}{x-a}$ ~~is~~ exists (is finite).

~~we~~ Since a is an arbitrary point $\in D$ we conclude that $f-g$ is differentiable on D, and

$$(f-g)' = f' - g' \qquad \square.$$

Excellent!

6. Show that if a function $f\colon D \to \mathbb{R}$ is differentiable at some $a \in D$, then f is also continuous at a.

This will hold if $\lim_{x \to a} (f(x) - f(a)) = 0$, since a is an accumulation point on f's domain. So we have:

$$\lim_{x \to a} (f(x) - f(a)) = \lim_{x \to a} \left[(f(x) - f(a)) \frac{(x-a)}{(x-a)} \right]$$

$$= \lim_{x \to a} \left[\frac{f(x) - f(a)}{(x-a)} \cdot (x-a) \right]$$

$$= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \cdot \lim_{x \to a} (x - a)$$

$$= f'(a) \cdot 0 = 0$$

since we know $f'(a)$ must be finite.

Excellent!

7. Let f:[a, b] → ℝ, and {x_n} be a sequence in [a, b] converging to c. Show $\lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right)$.

$\{x_n\} = \{\frac{1}{n}\}$ where $n \in \mathbb{N}$, a sequence in $(0,1] \subset [0,1]$ converging to 0.

$$f(x) = \begin{cases} x & \text{if } 0 < x \leq 1 \\ 1 & \text{if } x = 0 \end{cases}$$

counterexample

$\lim_{n\to\infty} f(x_n) = 0$

$f(\lim_{n\to\infty} x_n) = 1$

Excellent!

8. State and prove the Mean Value Theorem.

If

- f is continuous on $[a,b]$
- f is differentiable on (a,b)

then $\exists\ c \in [a,b) \ni f'(c) = \dfrac{f(b)-f(a)}{b-a}$

Proof: Let $\phi(x) = f(x) - f(a) - \dfrac{f(b)-f(a)}{b-a} \cdot (x-a)$.

Observe that $\phi(x)$ is continuous, and differentiable on $[a,b]$ and (a,b) respectively. Also note $\phi(a) = \phi(b)$.

We are obliged to use Rolle's Theorem,

Then $\exists\ c \in (a,b) \ni f'(c) - \dfrac{f(b)-f(a)}{b-a} = 0$.

Thus,

$$f'(c) = \frac{f(b)-f(a)}{b-a}. \quad \square$$

Nice.

9. State and prove (Bolzano's) Intermediate Value Theorem.

Bolzano's IVT: Let $f : [a,b] \to \mathbb{R}$ be a continuous function, and $f(a) \neq f(b)$. Then for every K between $f(a)$ and $f(b)$ there exists $c \in (a,b)$ such that $f(c) = K$.

Proof

~~Let~~ Assume $f(a) < f(b)$. The case $f(a) > f(b)$ is done similarly.

Let $X = \{x \in [a,b] \mid \cancel{x}\ f(x) < K\}$

Then X is non-empty for $a \in X$, and X is bounded above by b. Hence X has a supremum, call it c.

Let $\{x_n\}$ be a sequence in X converging to c,

and $\{t_n\}$ be a sequence in $[a,b] \setminus X$ converging to c.

Then $f(c) = f\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} (f(x_n))$ (f is continuous)

and $f(c) = f\left(\lim_{n\to\infty} t_n\right) = \lim_{n\to\infty} f(t_n)$

But $f(x_n) < K$, and $f(t_n) \geq K$

Hence by squeeze theorem, $f(c) = K$.

$\square$

Good

10. Suppose that f is a differentiable function from $\mathbb{R}$ to $\mathbb{R}$, and that g is a function from $\mathbb{R}$ to $\mathbb{R}$ for which $f \cdot g$ is differentiable on $\mathbb{R}$. What can you say about the differentiability of g?

Mostly nothing. It might be that g is differentiable (e.g. $f(x) = x$ and $g(x) = x$ gets $f \cdot g(x) = x^2$), but it might be g is not differentiable (e.g. $f(x) = x$ and $g(x) = |x|$ gets $f \cdot g$ differentiable, or for an extreme case $f(x) = 0$ and $g(x)$ Dirichlet's function, and $f \cdot g(x) = 0$ is differentiable).

But the best answer is that we can say g is differentiable at least where $f(x) \neq 0$. This is easily proved by applying the product rule to $(f \cdot g)(x)$ and $\left(\frac{1}{f}\right)(x)$, which is differentiable as long as $f(x) \neq 0$.