Exam 2b Calc 1 10/15/2015

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. State the formal definition of the derivative of a function $f(x)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Good

2. If $g(x) = \tan x + \arcsin x + x^3 + e^x + 7$, find $g'(x)$.

$$g(x) = \tan x + \arcsin x + x^3 + e^x + 7$$

Well,

SUM RULE $(f(x) + g(x))' = f'(x) + g'(x)$

So,

$$\boxed{g'(x) = \sec^2 x + \frac{1}{\sqrt{1-x^2}} + 3x^2 + e^x}$$

Because:

$(\tan x)' = \sec^2 x$ (because math).

$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$ (Proof on pg 6)

$(x^3)' = 3x^2$ because power rule.

$(e^x)' = e^x$ (because math).

$(k)' = 0$. (Proof on pg 2).

Excellent.

3. Use the definition of the derivative to find the derivative of $f(x) = x^2$.

$$f'(a) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$$

$$\lim_{h\to 0} \frac{x^2+2hx+h^2-x^2}{h}$$

$$\lim_{h\to 0} \frac{h(2x+h)}{h}$$

$$\lim_{h\to 0} 2x+h = 2x = f'(x)$$

Excellent!

4. Show that the derivative of $f(x) = k$ is $f'(x) = 0$.

If $f(x) = k$

then, $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h\to 0} \frac{k - k}{h}$$

$$= \lim_{h\to 0} \frac{0}{h} = \boxed{0}$$

Great!

5. Use the definition of the derivative to find the derivative of $f(x) = \sin x$.

Well $f'(x) = \lim_{h\to 0} \frac{f(x+h) \quad f(x)}{h}$ $f'(x) = \cos x$.

So $(\sin x)' = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$. Wonderful!

And $\sin(x+h) = \sin x \cosh + \cos x \sinh$

Then substitute.

$$= \lim_{h\to 0} \frac{\sin x \cosh + \cos x \sinh - \sin x}{h} = \lim_{h\to 0} \frac{\sin x \cosh - \sin x + \cos x \sinh}{h}.$$

(FACTOR) $= \lim_{h\to 0} \frac{\sin x(\cosh - 1) + \cos x \sinh}{h}$

(SUM RULE LIMITS) $= \lim_{h\to 0} \frac{\sin x(\cosh - 1)}{h} + \lim_{h\to 0} \frac{\cos x \sinh}{h}$.

(common mult. rule) $= \sin x \lim_{h\to 0} \frac{\cosh - 1}{h} + \cos x \lim_{h\to 0} \frac{\sinh}{h}$.

MATH says $\lim_{\theta\to 0} \frac{\cos\theta - 1}{\theta} = 0$ AND $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$

so, using substitution

$= \sin x \cdot 0 + \cos x \cdot 1 = \cos x$.

$\boxed{(\sin x)' = \cos x}$

6. State and prove the Product Rule for derivatives. Make it clear how you use any assumptions.

If f and g are differentiable functions, $(fg)'$ exists and $(f\cdot g)' = f'g + f\cdot g'$.

Proof: Well,

$$(f\cdot g)'(x) = \lim_{h\to 0} \frac{(f\cdot g)(x+h) - (f\cdot g)(x)}{h}.$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) \overbrace{- f(x)\cdot g(x+h) + f(x)\cdot g(x+h)}^{\text{Adding Zero}} - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \left[\frac{f(x+h) - f(x)}{h} \cdot g(x+h) + f(x) \cdot \frac{g(x+h) - g(x)}{h}\right]$$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \cdot g(x) + f(x) \cdot \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

We know this is $f'(x)$ since f is differentiable, and similarly with $g'(x)$.

$$= f'(x)\cdot g(x) + f(x)\cdot g'(x). \quad \square$$

7. Biff is a calculus student at Enormous State University, and he's having some trouble. Biff says "Crap. This Calculus stuff is pretty rough. So, like, the product rule I did okay, but now we're doing the chain rule and I'm pretty mixed up. Our professor was going on and on about how you've gotta know if it's f of g or g of f. But I was thinking it didn't even matter, since, like, if it's sine of x squared, that's the same whether you do the square and then sine or other way around. Like, with 0, sine 0 is 0, then square and it's 0, so it doesn't matter, right?"

   Help Biff by explaining as clearly as you can whether his reasoning holds, or if there are limitations

The example might be true but Say that $\theta = \pi/6$ + we take the Same equation $\sin \pi/6^2$. Taking the sin + then squareing you get 1/4. While Squareing first you get like .00478 wich is drasticly different. So in some cases you might get lucky with your idea, but it want work most of the time Just like $\sqrt{9} + \sqrt{9} = 6$ isnt equal to $\sqrt{9+9} = 4.24264$

Good

8. Show why the derivative of $y = \ln x$ is $y' = 1/x$.

$y = \ln x$

$e^y = e^{\ln x}$

$e^y = x$

$\frac{e^y \cdot y'}{e^y} = \frac{1}{e^y}$

$y' = \frac{1}{e^y}$

$y' = \frac{1}{e^{\ln x}}$

$\boxed{y' = \frac{1}{x}}$

(*) e an Ln always cross out to give
!

(i) Take the derivative.

(ii) divide by e^y because we always want to live y' by itself.

(iii) substitute y with ln x

Excellent!

9. a) What is the derivative of arcsin x?

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$$

b) Show why.

$$\arcsin x = y$$

$$\sin \arcsin x = \sin y$$

$$x = \sin y$$

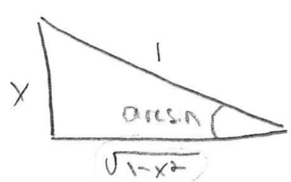
differentiating....

$$1 = \cos y \cdot 1 y'$$

$$1 = \cos y \cdot y'$$

$$\frac{1}{\cos y} = y'$$

$$\frac{1}{\cos(\arcsin x)} = y'$$



$$\boxed{\frac{1}{\sqrt{1-x^2}} = y'}$$

Excellent!

10. a) Find the derivative with respect to x of the curve $x^3 + y^3 = xy + 1$.

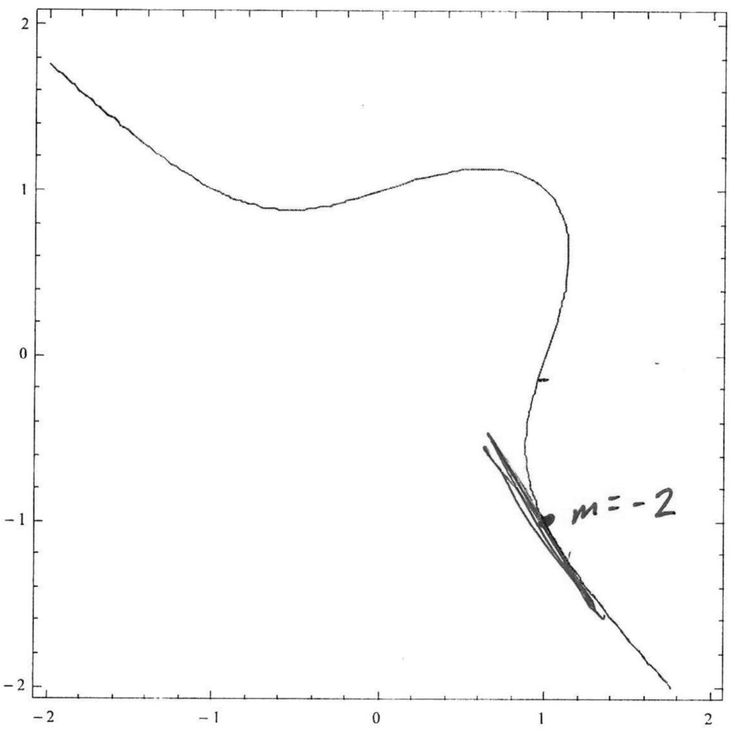
$$3x^2 + 3y^2 \cdot y' = 1 \cdot y + x \cdot y'$$

$$3y^2 \cdot y' - xy' = y - 3x^2$$

$$y'(3y^2 - x) = y - 3x^2$$

$$y' = \frac{y - 3x^2}{3y^2 - x}$$

b) Write an equation for the line tangent to the curve from part a at the point (1,–1).



At $(1,-1)$

$$y' = \frac{(-1) - 3(1)^2}{3(-1)^2 - (1)}$$

$$= \frac{-4}{2}$$

$$= -2$$

So using

$$y - y_0 = m(x - x_0)$$

we have

$$y - (-1) = (-2)(x - (1))$$

$$y + 1 = -2(x - 1)$$

or

$$y = -2x + 1$$