Exam 2b **Real Analysis 1** **11/3/16**

Each problem is worth 10 points. Show adequate justification for full credit. Don't panic.

1. a) State the definition of the derivative of a function $f(x)$ at $x = a$.

Let $f: D \to \mathbb{R}$ be a function with $D \subseteq \mathbb{R}$ and let $a \in D$ be an accumulation point of D. Then $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ provided $f'(a)$ is finite.

Great

b) State the definition of continuity of a function $f(x)$ at $x = a$.

Let $f: D \to \mathbb{R}$ with $D \subseteq \mathbb{R}$ be a function. We say f is continuous at a iff $\forall \varepsilon > 0$, $\exists \delta > 0 \ni x \in D$ and $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$.

Great

2. a) State the definition of a set E being closed.

A set E is closed iff every accumulation point of E is in E.

Good

b) State the definition of a set E being open.

A set E is open iff $\forall x \in E$, $\exists$ an open interval $I \subseteq E$ with $x \in I$.

3. a) State the Intermediate Value Theorem.

If $f: [a,b] \to \mathbb{R}$ is continuous and k is between $f(a)$ and $f(b)$, then $\exists c \in (a,b)$ for which $f(c) = k$.

b) State Rolle's Theorem.

If

- f is continuous on $[a,b]$
- f is differentiable on (a,b)
- $f(a) = f(b)$

then $\exists c \in (a,b) \ni f'(c) = 0$.

4. a) State the definition of a compact set.

A set is compact iff every open cover of that set has a finite subcover.

Good

b) State the Heine-Borel Theorem.

In $\mathbb{R}$, A set is compact iff it is closed and bounded.

Great

c) Give an example of a set with an open cover that has no finite subcover.

The set $(0,1)$ with an open cover of $(\frac{1}{n+1}, 1)$ $\forall n \in \mathbb{N}$

Excellent!

5. a) State the definition of uniform continuity.

Let $f: D \to \mathbb{R}$ with $D \subseteq \mathbb{R}$. We say f is uniformly continuous on $E \subseteq D \subseteq \mathbb{R}$ iff $\forall \varepsilon > 0$, $\exists \delta > 0 \ni \forall x, t \in E$, $|x-t| < \delta \to |f(x) - f(t)| < \varepsilon$.

Good

b) Give an example of a function which is continuous at every real number, but not uniformly continuous on $\mathbb{R}$.

$f(x) = x^2$ is continuous at every $a \in \mathbb{R}$,

but it is not uniformly continuous on $\mathbb{R}$.

↑ due to the δ values

6. State and prove the Product Rule for Derivatives.

If f and g are differentiable at a, then $(f \cdot g)(a)$ is <u>differentiable</u> and $(f \cdot g)'(a) = f'(a) \cdot g(a) + g'(a) \cdot f(a)$.

Well, $(f \cdot g)'(a) = \lim_{x \to a} \frac{(f \cdot g)(x) - (f \cdot g)(a)}{x - a} = \lim_{x \to a} \frac{f(x)g(x) - f(a)g(a)}{x - a}$. If we add 0 to this, we have $\lim_{x \to a} \frac{f(x)g(x) - f(a)g(x) + f(a)g(x) - f(a)g(a)}{x - a}$. We can simplify this to $\lim_{x \to a} g(x) \frac{f(x) - f(a)}{x - a} + f(a) \frac{g(x) - g(a)}{x - a}$. Using the <u>sum law</u> for limits, <u>product law</u> for limits, and <u>constant multiple</u> law for limits, we have $\lim_{x \to a} g(x) \cdot \lim_{x \to a} \frac{f(x) - f(a)}{x - a} + f(a) \lim_{x \to a} \frac{g(x) - g(a)}{x - a}$. Using the definition of derivative and applying the limits, we have $g(a) f'(a) + f(a) g'(a) = (f \cdot g)'(a)$. $\square$

well done

7. State and prove the Mean Value Theorem.

If f is continuous on $[a,b]$ and differentiable on (a,b), then $\exists\, c \in (a,b) \ni f'(c) = \dfrac{f(b)-f(a)}{b-a}$.

Proof: Let $\phi(x) = f(x) - f(a) - \dfrac{f(b)-f(a)}{b-a}(x-a)$. This function is continuous on $[a,b]$ by the rules for continuity, and differentiable on (a,b) by the rules for differentiability. Also, $\phi(a) = 0 = \phi(b)$, so by using Rolle's Theorem, we know $\exists\, c \in (a,b) \ni \phi'(c) = 0$.

But $\phi'(c) = f'(c) - \dfrac{f(b)-f(a)}{b-a}$ (using rules for derivatives), and

Since $\phi'(c) = 0$, $f'(c) - \dfrac{f(b)-f(a)}{b-a} = 0 \Rightarrow f'(c) = \dfrac{f(b)-f(a)}{b-a}$. $\square$

Excellent!

8. State and prove the Extreme Value Theorem.

B LUB ∃ g

If f is continuous on $[a,b]$, then it attains its maximum and minimum values on $[a,b]$.

Proof: Consider the maximum case. The minimum case follows similarly. Well, since f is continuous, f is bounded by the boundedness theorem. f has a least upper bound, call it M, by completeness. Now, suppose f does not achieve its maximum. Then there is no $c \in [a,b] \ni f(c) = M$. Let $g(x) = \frac{1}{M - f(x)}$. g is continuous, so it is bounded by the boundedness theorem. Call the bound K. Then $g(x) = \frac{1}{M-f(x)} \leq K$. Then $K(M - f(x)) \geq 1$ and $\frac{1}{K} \leq M - f(x)$ and $f(x) \leq M - \frac{1}{K}$. But this contradicts the fact that M was a least upper bound. Thus, $\exists\, c \in [a,b] \ni f(c) = M$.

Excellent

9. a) Prove or give a counterexample: If f' is bounded, then f is bounded.

Let $f(x) = x$. then $f'(x) = 1$, so it is bounded, but

x is not bounded.

Good

b) Prove or give a counterexample: If f is bounded, then f' is bounded.

Let $f(x) = \sin(x^2)$. then $f'(x) = 2x\cos(x^2)$.

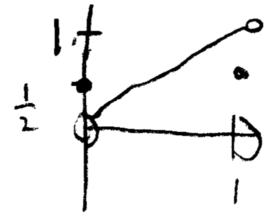
f is bounded, but f' is not bounded.

Excellent!

10. a) Prove or give a counterexample: If a function is defined on $[a, b]$ and continuous on (a, b), then it attains its maximum and minimum values on $[a, b]$.

Counter example: $f(x) = \begin{cases} \frac{1}{2} & \text{for } x = 0, 1 \\ x & \text{for } x \neq 0, 1 \end{cases}$

on $[0, 1]$



doesn't attain its maximums or minimums

Great

b) Prove or give a counterexample: If a function is continuous and bounded on $\mathbb{R}$, then it attains its maximum and minimum values.

$$\cancel{f(x) = \begin{cases} 5 - \frac{1}{x} & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -5 + \frac{1}{x} & \text{for } x < 0 \end{cases}}$$

never attains max of 5

Counterexample: $f(x) = \begin{cases} 5 - \frac{1}{x} & x > 1 \\ 4 & [-1, 1] \\ 3 + \frac{1}{x} & x < -1 \end{cases}$

Beautiful!