Exam 1 Calc 3 9/29/2017

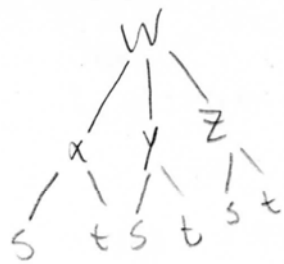
Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. State the formal definition of the partial derivative of a function $f(x, y)$ with respect to y.

$$f_y(x,y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x,y)}{h}$$

Good

2. Suppose that w is a function of x, y, and z, each of which is a function of s and t. Write the Chain Rule formula for $\frac{\partial w}{\partial t}$. Make very clear which derivatives are partials.



$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t}$$

They are all partial derivatives, because each is a function of multiple variables

Great

3. Find an equation for the plane tangent to $f(x, y) = \sqrt{xy}$ at the point (1, 1, 1).

$$f_x(x,y) = \frac{1}{2}(xy)^{-1/2} \cdot y = \frac{y}{2\sqrt{xy}}$$

$$f_y(x,y) = \frac{1}{2}(xy)^{-1/2} \cdot x = \frac{x}{2\sqrt{xy}}$$

$$f_x(1,1) = \frac{1}{2}$$

$$f_y(1,1) = \frac{1}{2}$$

So the tangent plane has equation:

$$z - 1 = \frac{1}{2}(x-1) + \frac{1}{2}(y-1)$$

4. Show that $\lim_{(x,y)\to(0,0)} \frac{x^4-4y^2}{x^2+2y^2}$ does not exist.

Along $x=0$

$$\lim_{(0,y)\to(0,0)} \frac{(0)^4-4y^2}{(0)^2+2y^2}$$

$$\lim_{y\to 0} \frac{-4\cancel{y^2}}{2\cancel{y^2}} = -\frac{4}{2}$$

Along $y=0$

$$\lim_{(x,0)\to(0,0)} \frac{x^4-4(0)^2}{x^2+2(0)^2}$$

$$\lim_{x\to 0} \frac{x^{\cancel{4}2}}{\cancel{x^2}}$$

$$\lim_{x\to 0} x^2 = 0$$

∴ the limit doesn't exist because when you approach it from two different direction you get two different limits

Well done

5. Let $f(x, y) = \dfrac{x}{x^2 + y^2}$. Find the maximum rate of change of f at the point (1,2) and the direction in which it occurs.

Maximum rate of change is in direction of gradient.

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle \Rightarrow \nabla f(1,2) = \left\langle \frac{3}{25}, -\frac{4}{25} \right\rangle$$

$$f_x(x,y) = \frac{1(x^2+y^2) - x(2x)}{(x^2+y^2)^2} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2}$$

$$f_y(x,y) = \frac{0(x^2+y^2) - x(2y)}{(x^2+y^2)^2} = -\frac{2xy}{(x^2+y^2)^2}$$

$$f_x(1,2) = \frac{(1)^2 + (2)^2 - 2(1)^2}{(1^2+2^2)^2} = \frac{3}{25}$$

$$f_y(1,2) = -\frac{2(1)(2)}{(1^2+2^2)^2} = -\frac{4}{25}$$

Excellent!

$$\sqrt{(3/25)^2 + (-4/25)^2} = \frac{1}{5}$$

Maximum rate of change of f at the point (1, 2) is $\frac{1}{5}$ and it occurs in the direction of $\left\langle \frac{3}{25}, -\frac{4}{25} \right\rangle$.

6. Show that for any vectors $\vec{a}$ and $\vec{b}$, the vector $\vec{a} \times \vec{b}$ is perpendicular to $\vec{a}$.

if $\vec{a} \cdot (\vec{a} \times \vec{b}) = 0$ then $\vec{a}$ is $\perp$ to $\vec{a} \times \vec{b}$

$$\vec{a} \times \vec{b} \quad \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} =$$

$$(a_2 b_3 i + b_1 a_3 j + a_1 b_2 k) - (b_2 a_3 i + a_1 b_3 j + b_1 a_2 k)$$

$$= \langle a_2 b_3 - b_2 a_3,\ b_1 a_3 - a_1 b_3,\ a_1 b_2 - b_1 a_2 \rangle$$

$$\vec{a} \cdot \nearrow \quad = a_1 a_2 b_3 - \boxed{a_1 b_2 a_3} + b_1 a_2 a_3 - a_1 a_2 b_3 + \boxed{a_1 b_2 a_3} - b_1 a_2 a_3$$

all cancel

$= 0$

B/C $\vec{a} \cdot (\vec{a} \times \vec{b}) = 0$ we know that $\vec{a}$ is $\perp$ to $\vec{a} \times \vec{b}$

0

Nice

7. Biff is a calculus student at Enormous State University, and he's having some trouble. Biff says "Crap, this Calc 3 stuff is killing me. There was this question on our practice exam about the direction derivatives, right? So like, if you know which direction is the level curve, and you know which direction is the gradient, and you face halfway in between them, then how big is the slope that way, right? So I obviously picked answer C, which said half as big as the gradient. But they said it was wrong, which is obviously because they're stupid, right?"

Explain clearly to Biff what can be said about the directional derivative in the direction halfway between the direction of a level curve and the direction of the gradient, and why.

We know that $D_u f = |\nabla f||\vec{u}| \cdot \cos\theta$, where θ is the angle between the gradient and the direction you travel. Since halfway between the gradient and the level curve is $\underline{\pm\pi/4}$, and $\underline{|\vec{u}| = 1}$ since $\vec{u}$ is a unit vector, then D_u in that direction isn't half the gradient, it's $\underline{\cos(\pm\pi/4) \cdot \nabla f}$.

because the level curve & the gradient are always perpendicular

Excellent!

(I'm running out of time, sorry for berevity) You nailed it.

8. Find the maximum and minimum values of $f(x, y) = 2x^2 + 3y^2 - 2x + 5$ subject to the constraint $x^2 + y^2 = 4$.

Langrange

$\nabla f = \langle 4x-2, 6y \rangle$

$\nabla g = \langle 2x, 2y \rangle$

$\nabla f = \lambda \nabla g$

$\langle 4x-2, 6y \rangle = \lambda \langle 2x, 2y \rangle$

$$\begin{cases} 4x-2 = 2\lambda x \\ 6y = 2\lambda y \\ x^2+y^2 = 4 \end{cases} \Rightarrow \begin{cases} 2x-1 = \lambda x & ① \\ 3y = \lambda y & ② \\ x^2+y^2 = 4 & ③ \end{cases}$$

② $\rightarrow$ $y=0$ or $\lambda = 3$

if $y=0$

$x^2+y^2=4$

$x^2=4$

$x=\pm 2$

if $\lambda=3$

$2x-1=3x$

$x=-1$

$x^2+y^2=4$

$y^2=3$

$y=\pm\sqrt{3}$

Excellent!

$(-2,0), (2,0), (-1,-\sqrt{3}), (-1,\sqrt{3})$

$f(-2,0) = 2\cdot 4 + 0 - 2\cdot(-2) + 5 = 8+0+4+5 = 17$

$f(2,0) = 2\cdot 4 + 0 - 2\cdot 2 + 5 = 8+0-4+5 = 9$

$f(-1,-\sqrt{3}) = 2 + 9 - 2\cdot(-1) + 5 = 2+9+2+5 = 18$

$f(-1,\sqrt{3}) = 2+9-2(-1)+5 = 18$

So the maximum is 18. the minimum is 9.

9. Find the maximum and minimum values of $f(x, y) = 2x^2 + 3y^2 - 2x + 5$ subject to the constraint $x^2 + y^2 \leq 4$.

We did the boundary circle in #8, so now just look inside:

$$4x - 2 = 0 \implies x = \frac{1}{2}$$
$$6y = 0 \implies y = 0$$

$$f\left(\frac{1}{2}, 0\right) = \frac{1}{2} + 0 - 1 + 5 = 4\frac{1}{2}$$

That's lower than the heights from #8 so $4\frac{1}{2}$ is our minimum, but 18 is still our maximum.

10. Find all critical points of $p(x, y) = 36xy\, e^{-2x-3y}$ and classify them as maxima, minima, or saddle points.

Take derivative

$$f_x = 36y \cdot e^{-2x-3y} + 36xy \cdot -2e^{-2x-3y} = (36y - 72xy)e^{-2x-3y}$$

$$f_y = 36x \cdot e^{-2x-3y} + 36xy \cdot -3e^{-2x-3y} = (36x - 108xy)e^{-2x-3y}$$

Set equal zero

$$0 = (36y - 72xy)e^{-2x-3y}$$

$$0 = (36x - 108xy)e^{-2x-3y}$$

* $0 = 36y \cdot (1 - 2x) \cdot$ (something that's never 0)

** $0 = 36x \cdot (1 - 3y) \cdot$ (something that's never 0)

* is true if $y = 0$ or $x = \frac{1}{2}$.

If $y = 0$, ** is only true for $x = 0$

If $x = \frac{1}{2}$, ** is only true for $y = \frac{1}{3}$

So $(0,0)$ and $(\frac{1}{2}, \frac{1}{3})$ are the critical points

$$f_{xx} = -72y \cdot e^{-2x-3y} + (36y - 72xy) \cdot -2e^{-2x-3y}$$

$$= (144xy - 144y)\, e^{-2x-3y}$$

$$f_{xy} = (36 - 72x)e^{-2x-3y} + (36y - 72xy) \cdot -3e^{-2x-3y}$$

$$= (36 - 72x - 108y + 216xy)e^{-2x-3y}$$

$$f_{yy} = -108x\, e^{-2x-3y} + (36\cancel{x} - 108xy) \cdot -3e^{-2x-3y}$$

$$= (-216x + 324xy) \cdot e^{-2x-3y}$$

$D(0,0) = (0 \cdot e^0)(0 \cdot e^0) - (36 \cdot e^0)^2 < 0 \quad \therefore$ saddle point

$D(\frac{1}{2}, \frac{1}{3}) = ((24 - 48)e^{-2})((-108 + 54)e^{-2}) - ((36 - 36 - 36 + 36)e^{-2})^2 > 0$

and $(24 - 48)e^{-2} < 0 \quad \therefore$ max