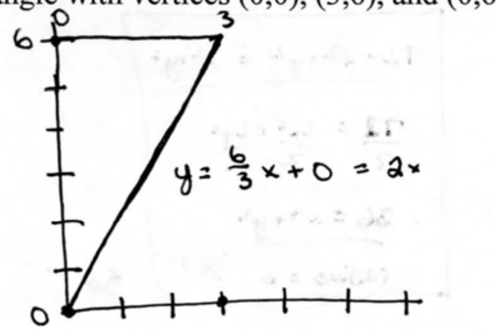
Exam 2 Calc 3 10/26/2017

Each problem is worth 10 points. For full credit provide complete justification for your answers. All integrals should be set up in terms of a single coordinate system, i.e., if you use cylindrical your integral should involve no *x* or *y*, etc.

1. Set up an iterated integral for the volume below $z = f(x, y)$ and above the *xy*-plane on the region *R*, a triangle with vertices (0,0), (3,6), and (0,6).



$$\int_0^3 \int_{2x}^6 \int_0^{f(x,y)} 1 \, dz\, dy\, dx$$

or

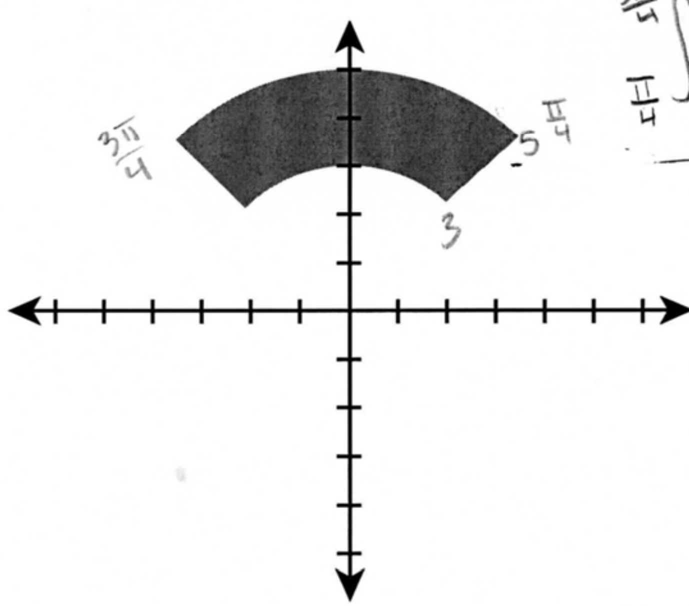
$$\int_0^3 \int_{2x}^6 f(x,y)\, dy\, dx$$

$m = \frac{\text{rise}}{\text{run}} = \frac{6}{3} = 2$

Good

2. Set up an iterated integral for the volume below $z = 7$ and above the *xy*-plane on the region *R* pictured below (the diagonal boundaries are the lines $y = x$ and $y = -x$):

$$\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_3^5 \int_0^7 1 \, r\, dz\, dr\, d\theta$$

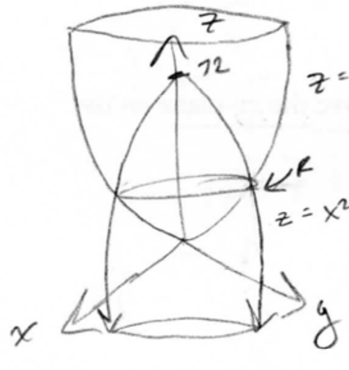


Great

3. Set up an iterated integral for the volume of the solid enclosed between the surface $z = x^2 + y^2$ and the surface $z = 72 - x^2 - y^2$.

$-$ paraboloid

$+$ paraboloid



$z = 72 - x^2 - y^2$

Nice

$z = x^2 + y^2$

$x^2 + y^2 = r^2$

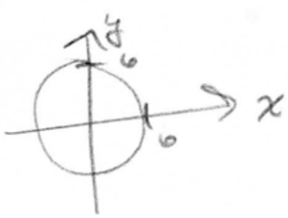
$72 - r^2 = r^2$

$72 = 2r^2$

$\sqrt{36} = \sqrt{r^2}$

$\pm 6 = r$

$6 = r$

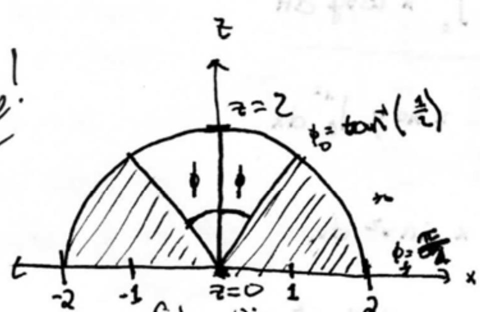


$$\int_0^{2\pi} \int_0^{6} \int_{r^2}^{72 - r^2} r \, dz \, dr \, d\theta$$

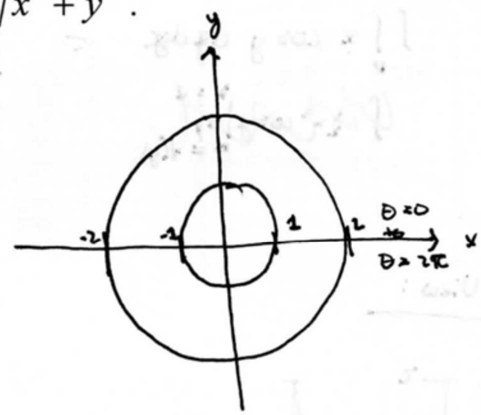
1

4. Set up an iterated integral for the volume of the solid lying within the sphere $x^2 + y^2 + z^2 = 4$, above the xy-plane, and outside the cone $z = 2\sqrt{x^2 + y^2}$.

Nice!



Side View



Top View

radius of 2 for sphere

at $z=0$ for cone

$0 = 2\sqrt{(x^2+y^2)}$

$0 = x^2+y^2$

$x=0 \quad y=0 \quad$ radius $=0$

at $z=2$ for cone

$2 = 2\sqrt{(x^2+y^2)}$

$\sqrt{(1)} = x^2+y^2$

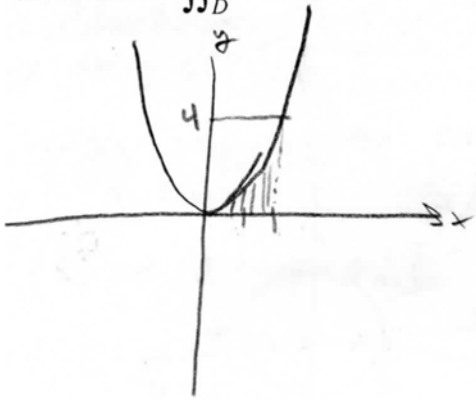
$1 = x^2+y^2 \qquad 1 = x^2+0$

radius $= 1 \qquad x = 1$

$$\int_0^{2\pi} \int_{\tan^{-1}(\frac{1}{2})}^{\frac{\pi}{2}} \int_0^2 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

Excellent!

5. Evaluate $\iint_D x\cos y\,dA$, where D is bounded by $y = 0$, $y = x^2$, and $x = 2$.



$$\int_0^2 \int_0^{x^2} x\cos y\, dy\, dx$$

$$= \int_0^2 x \cdot \sin y \Big|_0^{x} dx = \int_0^2 x \cdot \sin(x^2)\, dx$$

Let $u = x^2$

$du = 2x\, dx$

$\frac{du}{2} = x\, dx$

so then:

$$\frac{1}{2}\int_{x=0}^{x=2} \sin(u)\, du = -\frac{1}{2}\cos(x^2)\Big|_0^2$$

$$= -\frac{1}{2}\left(\cos(4) - 1\right)$$

$$= \frac{1}{2} - \frac{\cos(4)}{2}$$

Great

6. Compute the Jacobian for the conversion from rectangular to cylindrical coordinates.

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \\ \frac{\partial x}{\partial z} & \frac{\partial y}{\partial z} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

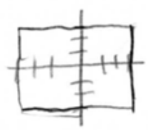
$$= r(\cos^2\theta + \sin^2\theta) \Rightarrow 1 \text{ by pyth theorem}$$

$$\boxed{= r}$$

Great

7. Bunny is a calculus student at Enormous State University, and she's having some trouble. Bunny says "Ohmygod, this Calc 3 stuff is just too much. I used to think symmetrical always made things easier, but now I'm really confused. I guess sometimes with the double integral thingies you can go from, like, – 3 to 3 both ways, or instead go from 0 to 3 and then times it by 4, right? But I think they were saying that you can't always. How do you tell when you can?

Give Bunny examples and explain why sometimes it would be okay in a double integral to use symmetry, and which times it wouldn't (**at least one** example each way).

It is going to depend on the integrand you are integrating over. The region you are using is a square going from -3 to 3 as shown here;  if the integrand you are using is symetric over this region, such as $z=1$, then you can say that it is equivalent to 4 multiplied by a double integral in a square from 0 to 3. However, if the integrand is not symetric across your bounds, such as $z=10+x+y$, the the integral across one 0 to 3 square does not necessarily equal the integrals across the other squares, and thus you cannot use symetry to evaluate the integral.

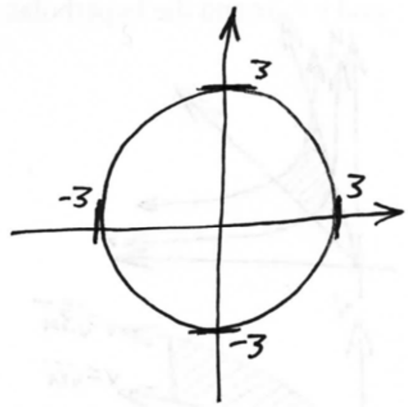
Great

8. Set up iterated integrals for the z-coordinate of the centroid of the solid bounded between the xy-plane and $z = 9 - x^2 - y^2$.

Intersection:

$$0 = 9 - x^2 - y^2$$

$$x^2 + y^2 = 9 \quad \text{Circle with radius } 3$$

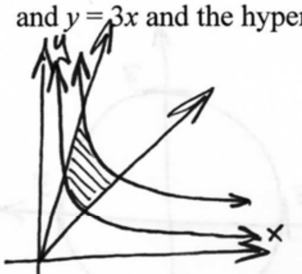


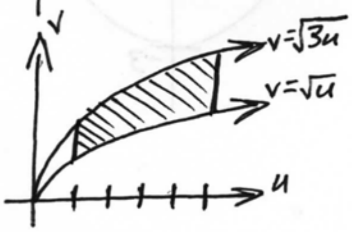
$$\boxed{\bar{z} = \frac{\iiint_E \rho(x,y,z)\, z\, dV}{\iiint_E \rho(x,y,z)\, dV}}$$

So here:

$$\bar{z} = \frac{\int_0^{2\pi}\int_0^{3}\int_0^{9-r^2} z \cdot k \cdot r\, dz\, dr\, d\theta}{\int_0^{2\pi}\int_0^{3}\int_0^{9-r^2} k \cdot r\, dz\, dr\, d\theta}$$

9. Evaluate $\iint_R xy\, dA$, where R is the region in the first quadrant bounded by the lines $y = x$ and $y = 3x$ and the hyperbolas $xy = 1$, $xy = 5$ by using the transformation $x = u/v$, $y = v$.





$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{v} & 0 \\ -\frac{u}{v^2} & 1 \end{vmatrix}$$

$$= \frac{1}{v} - 0$$

$$= \frac{1}{v}$$

$$(v) = \left(\frac{u}{v}\right) \Rightarrow u = v^2$$

$$(v) = 3\left(\frac{u}{v}\right) \Rightarrow u = \frac{1}{3}v^2$$

$$\left(\frac{u}{v}\right)(v) = 1 \Rightarrow u = 1$$

$$\left(\frac{u}{v}\right)(v) = 5 \Rightarrow u = 5$$

$$\iint_R xy\, dA = \int_1^5 \int_{\sqrt{u}}^{\sqrt{3u}} \left(\frac{u}{v}\right)(v) \cdot \frac{1}{v}\, dv\, du$$

$$= \int_1^5 \int_{\sqrt{u}}^{\sqrt{3u}} \frac{u}{v}\, dv\, du$$

$$= \int_1^5 u \cdot \ln v \Big|_{\sqrt{u}}^{\sqrt{3u}} du$$

$$= \int_1^5 \left(u \ln\sqrt{u} - u \ln\sqrt{3u}\right) du$$

$$= \int_1^5 \left(\frac{u}{2} \ln u - \frac{u}{2} \ln 3u\right) du$$

$$= \int_1^5 \frac{u}{2} \ln \frac{u}{3u}\, du$$

$$= \int_1^5 \frac{u}{2} \ln 3\, du$$

$$= \frac{u^2}{4} \ln 3 \Big|_1^5$$

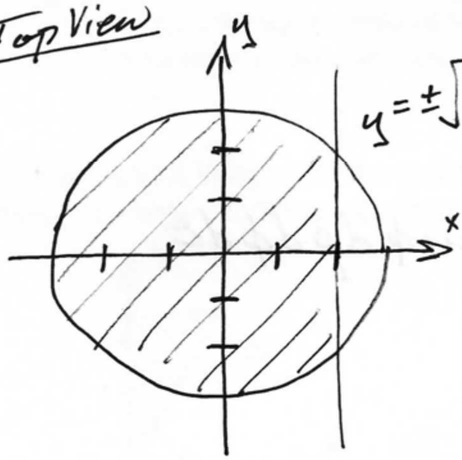
$$= \frac{25}{4} \ln 3 - \frac{1}{4} \ln 3$$

$$= \frac{24}{4} \ln 3$$

$$= 6 \ln 3$$

10. Consider the region under the surface $z = 18 - 2x^2 - 2y^2$, above the xy-plane, and with $x \le 2$. Set up an iterated integral for the volume of this solid.

Top View



$y = \pm\sqrt{9 - x^2}$

$0 = 18 - 2x^2 - 2y^2$

$2x^2 + 2y^2 = 18$

$x^2 + y^2 = 9$

$$\text{Volume} = \int_{-3}^{2} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{0}^{18-2x^2-2y^2} 1 \, dz\,dy\,dx$$