Exam 3 Calculus 3 12/1/2017

Each problem is worth 10 points. Show adequate justification for full credit. Please circle all answers and keep your work as legible as possible.

1. Parametrize and give bounds for a line segment from (–3, 0) to (1, 2).

$$\boxed{\begin{aligned} x(t) &= -3+4t \\ y(t) &= 0+2t \\ 0 \le\ & t \le 1 \end{aligned}}$$

Good

2. Let $\mathbf{F}(x, y, z) = \langle 6xy, 3x^2 + 6y^2 \rangle$. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the circle $x^2 + y^2 = 4$ with counterclockwise orientation.

Fun theorem of L.I.

$$3x^2y + 2y^3 \Big|_{(2,0)}^{(2,0)} = \boxed{0}$$

Same start and end points

Greens Theorem

$$\iint 6x - 6x \, dA$$

$$\iint 0 \, dA = \boxed{0}$$

Great

3. Let $\mathbf{F}(x, y, z) = \langle 3y, 2x, 5z \rangle$, and let S be the sphere with radius 1 centered at the origin with outward orientation. Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$.

divergence theorem

$\operatorname{div}\vec{F} = 0 + 0 + 5$

$= 5$

$\iiint 5 \, dA$

since it is a sphere volume $= 4/3\pi r^3$

$r = 1$

$5 \cdot \frac{4}{3}\pi 1^3$

$= 20/3\,\pi$

Great!

4. Compute $\int_C \vec{F} \cdot d\vec{r}$ for the vector field $\vec{F}(x,y) = xy\vec{i} - y\vec{j}$ and with C a line segment from (1, 2) to (3, –1).

$\langle xy, -y \rangle$ $f_{xy} = x$ $f_{yx} = 0$

NO potential function

NOT closed so long way.

Step 1. $\vec{r}(t) = \langle 1+2t, 2-3t \rangle$

$0 \leq t \leq 1$

$(1+2t)(2-3t)$

$2 + 4t - 3t - 6t^2$

Step 2. $\vec{F}(\vec{r}(t)) = \langle 2+t-6t^2, 3t-2 \rangle$

Step 3. $\vec{r}\,'(t) = \langle 2, -3 \rangle$

Step 4. $\int_0^1 \langle 2+t-6t^2, 3t-2 \rangle \cdot \langle 2, -3 \rangle \, dt$

Step 5. $\int_0^1 (4 + 2t - 12t^2 - 9t + 6) \, dt$

$\int_0^1 (-12t^2 - 7t + 10) \, dt$

$\left[-4t^3 - \frac{7t^2}{2} + 10t \right|_0^1$

$-\cancel{4}^{\,6} - \frac{7}{2} + 10$

$\frac{12}{2} - \frac{7}{2} = \frac{5}{2}$

Excellent

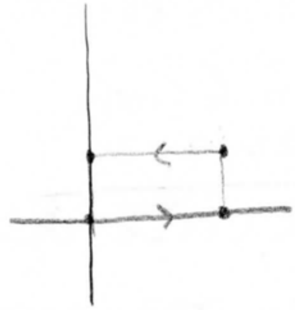
5. Let C be the positively oriented rectangle with vertices (0,0), (4,0), (4,3), (0,3). Evaluate the line integral $\int_C 9y^2x\,dx + x^2y\,dy$.

$P\,dx + Q\,dy$

$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$

Greens

$2xy - 18xy$



$$\int_0^4\int_0^3 (2xy - 18xy)\,dy\,dx$$

$$\int_0^4\int_0^3 (-16xy)\,dy\,dx$$

$$\int_0^4 \left(-8xy^2 \Big|_0^3\right) dx$$

$$\int_0^4 (-8 \cdot 9x)\,dx$$

$$-4 \cdot 9x^2 \Big|_0^4$$

$$\boxed{-4 \cdot 9 \cdot 16}$$

Great!

6. Prove that if **F**(*x*,*y*,*z*) is a vector field whose component functions have continuous second-order partial derivatives, then div(curl **F**) = 0. **Make it clear how the requirement that the partials be continuous is important.**

$\vec{F} = \langle P, Q, R \rangle$ Let's start with curl $\vec{F}$:

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ P & Q & R \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \end{vmatrix} = \langle \underline{Q_z - R_y}, \underline{R_x - P_z}, \underline{P_y - Q_x} \rangle$$

Now let's take the div of that:

$$\langle \partial/\partial x, \partial/\partial y, \partial/\partial z \rangle \cdot \langle Q_z - R_y, R_x - P_z, P_y - Q_x \rangle =$$

$$Q_{zx} - R_{yx} + R_{xy} - P_{zy} + P_{yz} - Q_{xz}$$

By Clairaut's Theorem, if $\vec{F}$ has continuous second order partial derivatives:

$$\underline{Q_{zx} = Q_{xz}} \quad \underline{R_{yx} = R_{xy}} \quad \underline{P_{zy} = P_{yz}}$$

So let's rearrange for clarity:

$$\underbrace{(Q_{zx} - Q_{xz}) + (R_{xy} - R_{yx}) + (P_{yz} - P_{zy})}_{\text{all 3 of these cancel because of Clairaut's Theorem.}} = 0$$

'0

well done

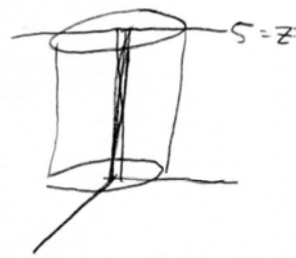
7. Biff is a calc 3 student at a large state university and he's having some trouble. Biff says "Well crap, I thought I was good at math until Calc 3. I'm pretty good at getting answers, you know? But now we gotta say why there's no answers, which is pretty stupid. This one on our test was to show why there's no potential function for $\langle -y, x\rangle$, which is crazy, 'cause probably you just haven't looked hard enough, right? I mean, doesn't there gotta be an answer?"

   Explain as clearly as possible to Biff what you can say about the existence of a potential function for his vector field, and why.

   So Biff, you have to think about it from the beginning, okay? Like, if there _was_ a potential function $f(x,y)$, it would be one where its gradient $\langle f_x, f_y\rangle$ came out to $\langle -y, x\rangle$, right? But we also know Clairaut's Theorem promises the mixed partials f_{xy} and f_{yx} are equal, right? So if $f_x = -y$ and $f_y = x$, we do partials of those and get $f_{xy} = -1$ and $f_{yx} = 1$. But those aren't equal, so there never was any such f. It's not about looking hard enough if you're looking for something that doesn't exist.

8. Let $\mathbf{F}$ be the vector field $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + 2\,\mathbf{k}$. Let S be the portion of the cylinder $x^2 + y^2 = 9$ between $z = 0$ and $z = 5$, oriented away from the z-axis. Find $\iint_S \mathbf{F} \cdot d\mathbf{S}$.

Long way!



I) $\vec{r}(u,v) = \langle 3\cos u, 3\sin u, v \rangle$

$0 \le u \le 2\pi \qquad 0 \le v \le 5$

II) $\vec{F}(\vec{r}(u,v)) = \langle 3\cos u, 3\sin u, 2 \rangle$

III) $\vec{r}_u = \langle -3\sin u, 3\cos u, 0 \rangle$

$\vec{r}_v = \langle 0, 0, 1 \rangle$

$\vec{r}_u \times \vec{r}_v = \langle 3\cos u, +3\sin u, 0 \rangle$

IV) $\vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v) = 9\cos^2 u + 9\sin^2 u + 0 = 9$

V) $\int_0^{2\pi} \int_0^5 9 \, dv\, du = 90\pi$

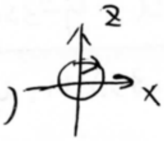
Excellent

9. Let $\mathbf{F}(x, y, z) = y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$, and let S be the hemisphere $x^2 + y^2 + z^2 = 1$, $y \geq 0$, oriented in the direction of the positive y-axis. Evaluate $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$.

Stoke's Thm. $\iint_S \text{curl}\,F \cdot ds = \int_C \vec{F} \cdot d\vec{r}$

The cross section on x-z plane is $x^2+y^2=1$

so $x = \sin t$, $y = 0$, $z = \cos t$. (because clockwise)



$\vec{r} = \langle \sin t, 0, \cos t \rangle$ $\quad 0 \leq t \leq 2\pi$ $\qquad \vec{r}\,' = \langle \cos t, 0, -\sin t \rangle$

$\vec{F}(\vec{r}) = \langle 0, \cos t, \sin t \rangle$

$2\cos^2 t - 1 = \cos 2t$

$\cos^2 t = \frac{1}{2}\cos 2t + \frac{1}{2}$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \langle 0, \cos t, \sin t \rangle \cdot \langle \cos t, 0, -\sin t \rangle \, dt$$

$$= \int_0^{2\pi} -\sin^2 t \, dt$$

$$= \int_0^{2\pi} \cos^2 t - 1 \, dt$$

$$= \int_0^{2\pi} \frac{1}{2}\cos 2t - \frac{1}{2} \, dt$$

$$= \frac{1}{4}\sin 2t - \frac{1}{2}t \Big|_0^{2\pi}$$

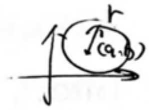
$$= -\pi$$

Nice Job!

$\langle 5b+5r\sin t, 5a+5r\cos t \rangle$

10. Let $\mathbf{F}(x, y) = \langle 5y, 5x \rangle$. Find the radii and centers of all circles C such that $\int_C \mathbf{F} \cdot d\mathbf{r} = 9$.

Green's Thm.



Suppose the center of ~~cir~~ circle is (a,b) and radius is r.

$$\int_C \vec{F} \cdot dr = \iint_R 5-5\,dA = 0.$$

$x = a + r\cos t$, $y = b + r\sin t$

There's no circle that can let $\int_C \vec{F} \cdot dr = 9$.

Nice reasoning ✓

10. Let $\mathbf{F}(x, y) = \langle 5y, 5x \rangle$. Find the radii and centers of all circles C such that $\int_C \mathbf{F} \cdot d\mathbf{r} = 9$.

F.T.L.I. sees a potential function $f(x,y) = 5xy$.

Well, since there is a potential function and our path is a circle, our endpoints will be the same. Let's call it some arbitrary point (a,b)

For any path, then, (including circles of any radius, centered at any point starting & ending at (a,b)) the Fun. Theorem for Line Integrals can be used to evaluate:

$$\int_C \vec{F} \cdot d\vec{r} = f(a,b) - f(a,b) = 0$$

This will always evaluate to 0 since our endpoints are identical, so no circles exist that will evaluate to 9.

Nice reasoning!