Exam 3 Calc 1 10/26/2018

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. What is $(\ln x)'$?

$(\ln x)' = \frac{1}{x}$ Good

let $y = \ln x$

$e^y = e^{\ln x}$

$e^y = x$

$e^y \cdot y' = 1$

$y' = \frac{1}{e^y}$

$y' = \frac{1}{e^{\ln x}}$

$y' = \frac{1}{x}$

2. a) What is $\left(xe^x\right)'$?

$1(e^x) + x(e^x)$

$(xe^x)' = e^x + xe^x$

b) What is $\left(x \arcsin x\right)'$? $\frac{1}{\sqrt{1-x^2}}$

$(1)(\arcsin x) + (x)\left(\frac{1}{\sqrt{1-x^2}}\right)$

$(x \arcsin x)' = \arcsin x + \frac{x}{\sqrt{1-x^2}}$

c) What is $\left(x \cosh x\right)'$?

Great

$(1)\cosh x + x(\sinh x)$

$\cosh x + x\sinh x = (x\cosh x)'$

3. Evaluate $\lim_{x\to\infty}\frac{x}{e^x}$. Be sure to provide good justifications for your steps.

$$\underbrace{\lim_{x\to\infty}\frac{x}{e^x}}\overset{L'H}{=}\lim_{x\to\infty}\frac{1}{e^x}=0$$

It's an $\frac{\infty}{\infty}$ indeterminate form, so we can use L'Hôpital's Rule!

$f'g+g'f$

4. Differentiate $y=\overset{\text{product}}{x\cos^{-1}x}-\overset{\text{chain}}{\sqrt{1-x^2}}$. [Hint: $\left(\cos^{-1}x\right)'=\frac{-1}{\sqrt{1-x^2}}$]

$$y=x\cos^{-1}x-(1-x^2)^{1/2}$$

$$y'=(x)'(\cos^{-1}x)+\left(\frac{-1}{\sqrt{1-x^2}}\cdot x\right)-\frac{1}{\cancel{2}}(1-x^2)^{-1/2}(-\cancel{2}x)$$

$$y'=(\cos^{-1}x)-\frac{x}{\sqrt{1-x^2}}+\frac{x}{\sqrt{1-x^2}}$$

$$\boxed{y'=\cos^{-1}x}$$

Excellent!

5. [Stewart] The table below gives estimates of the world population, in millions, from 1750 to 2000:

Year	Population	Year	Population
1750	790	1900	1650
1800	980	1950	2560
1850	1260	2000	6080

Use the exponential model and the population figures for 1800 and 1850 to predict the world population in 1900. Compare with the actual population.

$P(t) = ae^{kt}$

$1{,}260 = 980e^{50k}$

$1.286 = e^{50k}$

$\ln 1.286 = \ln e^{50k}$

$\frac{\ln 1.286}{50} = \frac{50k}{50}$

$0.00503 = k$

$P(t) = 1{,}260e^{50(0.0503)}$

$P(t) = 1{,}260(1.285714)$

$\boxed{P(t) = 1{,}620 \text{ million people}}$

The actual population in 1900 was 1,650 million people which means the rate of change in the world's population increased.

Good

6. Show why $(a^x)' = (\ln a)a^x$.

$y = a^x$

$\ln y = \ln a^x$ (take logs of both sides)

$\ln y = x \ln a$ (move x to front)

$\frac{1}{y} \, dy/dx = \ln a$ (differentiate)

$dy/dx = \ln a \cdot y$ (solve for dy)

$dy/dx = \ln a \cdot (a^x)$ (plug in y value to un-substitute)

$(a^x)' = (\ln a)a^x$

Great!

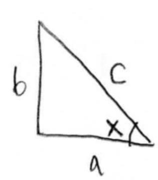
$\frac{1}{\sin x}$ $\sin^{-1} x$ $\arcsin x$ $\csc x$

7. Biff is a calculus student at Enormous State University, and he's having some trouble. Biff says "Geez, calculus is hard! All this ticky-tack stuff is just literally killing me. Like, you know, sometimes they write 1 over sin, and sometimes they write $\sin^{-1}$, and sometimes they write arcsin, and sometimes they write csc, and I think maybe they're all the same, but who the heck knows? I bet literally nobody actually can tell which ones are different. "

Explain clearly to Biff which of the functions he describes are actually the same, and which are different, and why.

There is a clear difference between inverse trigonometric functions and opposite trigonometric functions. When any trig function, sin, cos, tan, etc, is written with a negative exponent, you would think that means it is in the denominator of a fraction like $x^{-1} = \frac{1}{x}$ but it isn't. that would be $[\sin x]^{-1}$. $\sin^{-1} x$ is another way of writing $\arcsin x$ because all of those letters can be a bit much sometimes for mathematicians. ☺ The opposite function $\csc x = \frac{1}{\sin x}$ because:

Excellent!



$\sin x = \frac{opp}{hyp}$ or $\frac{b}{c}$

and

$\csc x = \frac{hyp}{opp}$ or $\frac{c}{b}$

They are reciprocals (or opposites), like how $\frac{1}{2}$ is to $\frac{2}{1}$.

8. Show that if $g(x) = \tan^{-1} x$ then $g'(x) = \dfrac{1}{1+x^2}$.

$$y = \tan^{-1} x$$

$$\tan y = \tan(\tan^{-1} x)$$

$$\tan y = x$$

Differentiate: $\sec^2 y \cdot y' = 1$

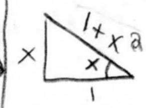
Evaluate: $1+x^2\, y' = 1$

Nice!

$$\boxed{y' = \frac{1}{1+x^2}}$$

Trig break! ☺

$$\tan y = \frac{x}{1} = \frac{opp}{adj}$$



$$\sec y = \frac{1}{\cos y}$$

$$\cos y = \frac{adj}{hyp}$$

therefore

$$\sec y = \frac{hyp}{adj} = \frac{1+x^2}{1}$$

9. Why is $\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}$?

$y = \sinh^{-1} x$

$\rightarrow \sinh y = x$ Swith to inverse function

$y' \cosh y = 1$ Differentiate implicitly

$y' = \frac{1}{\cosh y}$ Solve for y'

$y' = \frac{1}{\sqrt{1 + \sinh^2 y}}$ Use the identity $\cosh^2 x - \sinh^2 x = 1$.

$y' = \frac{1}{\sqrt{1 + \sinh(\sinh^{-1}(x))^2}}$

Well done

$\boxed{y' = \frac{1}{\sqrt{1+x^2}}}$

10. Evaluate $\lim_{x \to 0^+} x \ln x$. Be sure to provide good justifications for your steps.

$\lim_{x \to 0^+} \dfrac{\ln x \to -\infty}{\left(\frac{1}{x}\right) \to \infty}$ Write as a fraction.

This is in $\frac{(-)\infty}{\infty}$ indeterminate form, so

we can use L'Hôpital's Rule which he stole

from Bernoulli.

$$\underset{(\text{Ber.})}{\overset{\text{L'H}}{=}} \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \to 0^+} \frac{1}{x} \div \frac{-x^2}{1} = \lim_{x \to 0^+} -x = \boxed{0}$$

derivatives of top + bottom

Simplify

Excellent!